KEY

TO

PLANE AND SOLID GEOMETRY

DURELL – ARNOLD

KEY

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PLANE AND SOLID GEOMETRY

BY

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KEY TO PLANE GEOMETRY

PAGE 11

17. He first sets out two of the trees in position and then sights through them and places the other trees in line with the two already set out.

22. Straight, *I*; curved, *C, O, Q, S, U*; broken, *A, E, F, H, K, L, M, N, T, V, W, X, Y, Z*; mixed, *B, D, G, J, P, R*.

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3. 47°; 137°.

4. 32° 41′; 27° 36′ 17″; 122° 41′; 117° 36′ 17″.

5. Construct a right angle having the same vertex as the given angle and a side in common with this angle, and including the given angle as a part of itself.
To construct the supplement, produce a side of the given angle through the vertex.

6. Obtuse.

7. 6°.

8. 60°; 90°; 150°.

9. 135°; 22° 30′; 7° 30′.

10. 10 min.; $8\frac{1}{3}$ min.; 40 min.; 2 hr.; 1 hr. 40 min.; 8 hr.

11. 810°.

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12. $\frac{2}{5}$.

13. 30°; 90°; $22\frac{1}{2}$°; 15°; $1\frac{7}{8}$°.

14. 1 lb.; $\frac{3}{5}$ lb.; $1\frac{1}{2}$ lb.; $2\frac{1}{3}$ lb.; $\frac{1}{10}$ lb.; 6 lb.

15. Acute; obtuse; right.

18. Obtuse; acute.

19. Reflex; acute.

20. Right; obtuse; acute.

21. Right angle, for, denote the given angle by x. Then $(180° - x) - (90° - x) = 90°$.

22. (*a*) $r = t$. (*b*) $r = t$. (*c*) $t > r$, for s and r are acute. $\therefore$ t is obtuse. (*d*) $r > t$, for s is acute and $\therefore$ r is obtuse.

23. (*a*) $m > p$. (*b*) $m = p$. (*c*) $m < p$. (*d*) Denote the comp. of m by x. Then $x + m = 90°$, $x + p = 180°$. $\therefore$ $p - m = 90°$. $\therefore$ $p = m + 90°$. $\therefore$ $p > m$.

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24. $x = 2\,(90° - x)$. $\therefore\ x = 60°$. *Ans.*

25. $x = \frac{1}{3}(180° - x)$. $\therefore\ x = 45°$. *Ans.*

26. $x = \frac{4}{5}\,(180° - x)$. $\therefore\ x = 80°$. *Ans.*

27. (1) $x = 90° - x + 12°$. $\therefore\ x = 51°$. *Ans.*
(2) $x = 180° - x + 15\frac{1}{2}°$. $\therefore\ x = 97°\,45'$. *Ans.*

28. $90° - x + 180° - x = 126°$. $\therefore\ x = 72°$. *Ans.*

29. $180° - x = 4(90° - x)$. $\therefore\ x = 60°$. *Ans.*

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2. Yes, by placing the edge of the ruler on the pipe in a direction parallel to the length of the pipe. No.

3. No.

4. Place the straight edge on the surface in various directions. In all positions every point of the straight edge should be in contact with the surface of the tennis court.

5. Yes.

PAGE 24

1. Ax. 7.

2. 140°; 210°; Ax. 7.

3.
$$\begin{array}{r@{\ =\ }r} 7 & 7 \\ -\,2 & -\,2 \\ \hline 5 & 5 \end{array}$$

4. (1)
$$\begin{array}{r@{\ =\ }r} 5 & 5 \\ \times 2 & \times 2 \\ \hline 10 & 10 \end{array}$$

(2) $12 = 12$. Dividing each of these by 3, $4 = 4$.

5. $8 = 8$. $\therefore\ \sqrt[3]{8} = \sqrt[3]{8}$, or $2 = 2$.

6. Ax. 4.

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7. Ax. 2.

8. Use the diagram of Ex. 7. Thus, if $LN = MO$, then $LM = NO$ (Ax. 3).

11. Ax. 1.

12. Ax. 2.

13. Ax. 9.

14. Through two points only one straight line can be passed; and a right angle is half of a straight angle.

PAGE 28

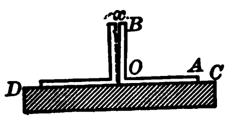
1. Three.
2. Six.
3. In Ex. 1 each point helps to locate two lines; in Ex. 2, each point helps to locate three lines. Hence, a point in the latter case does $1\frac{1}{2}$ times as much work, or is $1\frac{1}{2}$ times as efficient as in the former case.

5. 140.° 6. $r = 43°$, etc.

8. $\angle ABP = 40°$; $\angle PBQ = 50°$; $\angle QBC = 40°$.

9. $\angle COD = 180° - 142° = 38°$. $\angle COB = 120° - 38° = 82°$. *Ans.*

10.



Place AO, one of the inside edges of the square, in contact with the straight edge CD. Then repeat the act, placing O at the same point on CD, with OA pointed in the opposite direction from the first position.
By § 23, the sum of two right angles must be a straight angle.

PAGE 30

2. Any two of the angles which are not vertical.
3. 157° (§ 69).
4. $s = p = 153° - 107° = 46°$. *Ans.*
5. $t = q = 138°$ (see Ex. 9, p. 28).
6. $4r = 180° - 24°$. $\therefore r = x = 39°$. *Ans.*

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2. 72°; 104°.
3. Add the complement to 90°; subtract 90° from the supplement.
4. 150°; 12° 30′.
5. As the sides of a triangle; two || lines, one through each point, and a third line through the two given points; etc.

6. 12 in. 7. No. 8. No.

10. Denote the angle of the square by y. Then
$2(y + e) = 2y + x$. $\therefore 2e = x$. $\therefore e = \frac{1}{2}x$.

PAGE 33

1. $EF = 18$; $\angle E = 70°$; $\angle F = 50°$.

PAGE 35

1. $DE = 24$; $EF = 27$; $\angle E = 55°$.

1. (Group 9). Use § 79.

2. § 79. 3. § 79. 4. § 79.

5. Prove $\triangle DCF = \triangle ACB$ by § 79. 217 yd.

6. § 80.

PAGE 36

7. § 80. 8. § 80.

9. $\angle ABO = \angle CBO$ by § 66. Use § 80.

10. Use § 80. 137 yd.

PAGE 37

1. By the principle proved in § 82, $\angle C = \angle A = 67°$.

PAGE 39

1. $\angle BAC = \angle BCA$ (§ 82). $\therefore p = r$ (§ 66).
Then prove $\triangle DAB = \triangle BCF$ by § 79.

2. Use § 82 twice and Ax. 3.

3. § 83. 4. § 83.

5. Draw AC and use § 83.

PAGE 41

2. Construct a right angle and bisect it. Also through the vertex draw a line $\perp$ the bisector.

PAGE 50

2. No, for $d = 110°$, and hence d and f are not supplementary. Use § 99.

1. (Group 11). $\angle MOQ = \angle POL$ (§ 69). $PO = OQ$ (Hyp.).
$\angle MQO = \angle OPL$ (§ 96). $\therefore \triangle POL = \triangle OMQ$ (§ 80).

2. $\angle A = \angle C$ (§ 82); $\angle D = \angle C$ (§ 96), etc.

3. Prove $\triangle ABP = \triangle PCD$ by § 80, etc.

4. $x = p$ (§ 69); $y = q$ (§ 69); $x = y$ (Ax. 1). $\therefore AB \parallel CD$ (§ 89).

5. $\angle ABC = \angle BCD$ (Ax. 2), etc.

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6. $y = m$ (§ 82). $x = y$ (§ 97); $l = m$ (§ 97), etc.
7. Prove $\triangle ABD = \triangle BFC$ by § 79, etc.
8. $\angle B = \angle C$ (§ 82). Then use § 96 twice and Ax. 1.
9. $\angle CBE = 115°$ (Ax. 7). $\therefore$ $\angle ABE = 65°$ (§ 32).
   $\therefore$ $BE \parallel CD$ (§ 91).
10. $\angle B = \angle i$ (step 6, p. 43), etc.

PAGE 53

1. 62°.
2. 53° 45′.
3. 60°.
4. 45°.
5. No.

PAGE 54

6. 138°.
7. 142°; 115°; 103°.
8. 71°.
9. 80°.
10. 78°; 78°; 24°.
11. Construct an equilateral triangle and bisect one of its angles. Bisect an angle of 30°.
12. Construct the supplement of 60°. $75° = 45° + 30°$.
13. $150° = 90° + 60°$. $195° = 180° + 15°$.
15. Construct an equilateral triangle and a perpendicular to the base through an extremity of the base.
16. See Ex. 12.
17. Construct an angle of 45° at each end of the 2-in. line.
19. Corr. ∠s are = (§ 107). △s are not equal.
20. Through the vertex of the acute angle construct a $\perp$ to one side of the angle.
21. Produce one side of the angle through the vertex.

PAGE 57

1. Use § 114.
2. § 114.
3. $\angle B$ (§ 114).
4. § 114.
5. $\angle OAQ = \angle OBP$ (§ 114).
   $\angle AOQ = \angle BOP$ (§ 69).
   $\angle AOB = \angle POQ$ (§ 69).
   $\angle AQO = \angle OQC = \angle APC = \angle APB$ (§ 63).
6. See Ex. 1, p. 39.

PAGE 60

1. Use § 117. $PQ = PR$ (corr. sides of $= \triangle$).

PAGE 61

1. Use § 110.

PAGE 63

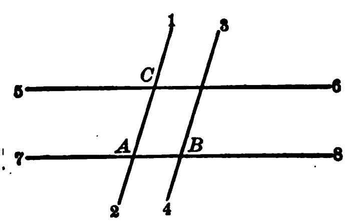
4. The locus is a straight line, parallel to the top of the level track and 1 ft. above it.
5. The locus is a straight line $\perp$ the line joining the two given points at the midpoint of this line.
6. Last sentence of § 123.

PAGE 69

1. Ineq. Ax. 2.
2. (1) $\begin{array}{r} 7 > 5 \\ +3+3 \\ \hline 10 > 8 \end{array}$ (2) $\begin{array}{r} 7 > 5 \\ -2-2 \\ \hline 5 > 3 \end{array}$ (3) $\begin{array}{r} 7 > 5 \\ 5 > 3 \\ \hline 12 > 8 \end{array}$
3. (1) See Ex. 1. (2) $10 > 8$. $\therefore 5 > 4$ Ill. of Ineq. Ax. 3, $\begin{array}{rr} 10 = & 10 \\ -6 > & -2 \\ \hline 4 < & 8 \end{array}$
4. $9 > 7 > 3$. $\therefore 9 > 3$. 5. § 134.

PAGE 72

1. Four; five. Three points will not determine a pair of parallel lines, for two pairs might be passed through the three points; thus through the points A, B, C the $\|$ lines 12 and 34 might be passed, also the parallel lines 56 and 78.



2. No.

PAGE 74

1. Hence, $\angle APB$ is obtuse. Use § 138.
2. $AB + BF > AF$ (§ 78). Also $FC + CD > FD$ (§ 78). $AD = AD$ (Ident.). Adding by Ineq. Ax. 1, $AB + BC + CD + DA > AF + FD + AD$.

3. $BD + DF > DF$ (§ 78). To each of these unequals add $AD + FC$ (Ineq. Ax. 1).

4. Use $\triangle$s ABD and ADC, and § 138.

5. Use $\triangle$s ABD and DBC, and § 139.

PAGE 76

2. Yes; no.

3. $\angle$s of a square are rt. $\angle$s; $\angle$s of a rhombus are oblique. Have same number of sides; sides are equal; opposite sides are parallel.

4. 48 in.

5. Rhombus, rhomboid, parallelogram, quadrilateral.

6. Have same number of sides; opposite sides are parallel.

7. Two $\triangle$s; isosceles.

9. Construct a rectangle and a rhomboid whose corresponding sides are equal.

PAGE 77

1. $\angle r = \angle s$ and $\angle q = \angle p$ (corr. $\angle$s of $=$ $\triangle$s).
   $\therefore\ \angle BAD = \angle BCD$ (Ax. 2).

2. Produce BC to F and DC to H.
   Then $\angle BAD = \angle HCF$ (§ 112), $\angle BCD = \angle HCF$ (§ 69).
   $\therefore\ \angle BAD = \angle BCD$ (Ax. 1).

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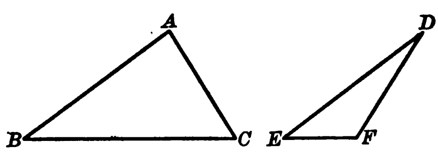
1. Four, viz.: AFD and BFC; AFB and DFC; ABD and CBD; ABC and ADC.

2. $180° - p° - q°$. 3. $90° - 2x°$.

7.



Construct the triangles ABC and DEF in which $\angle B = \angle E$, $AB = DE$, and $AC = DF$. See text-book, p. 159.

PAGE 79

1. Geom. Ax. 2, Post. 1 (§ 46), § 82 (twice), Ax. 2 (§ 41), Ax. 7 (§ 41), § 79.

PAGE 80

1. Use § 98.
2. 43°; 137°.
3. Denote the given angles by x and $3x$.
   Then $x + 3x = 180°$ (§ 98). $\therefore\ x = 45°$; $3x = 135°$. *Ans.*
4. $a°$; $180° - a°$.
5. AC, longest; AB, shortest (§ 135).

PAGE 81

1. § 159.
2. No. (1) Converse of § 69, viz.: if two angles are equal, they are vertical. (2) Converse of § 112, viz.: if two angles are equal, their corresponding sides are parallel.
3. If we know that the diagonals of a quadrilateral bisect each other, by § 162, we at once know without effort that the given quadrilateral is a parallelogram.

PAGE 82

1. $AD \parallel BC$ (§ 92). Then use § 161. AB is 312 yd. (§ 155).
2. $BH = CF$ (Ax. 2). Then prove $\triangle ABH = \triangle DFC$ by § 79.
3. In $\triangle$ DPC and AQB, $DC = AB$ (§ 155), $DP = BQ$ (Hyp.), $\angle PDC = \angle QBA$ (§ 96). Use § 79.
4. Prove $\triangle RAB = \triangle DCS$ by § 79. Also $\triangle RAD = \triangle BCS$ by § 79, etc. Use § 160.
5. In $\triangle$ AOP and QOC, $AO = OC$ (§ 159), $\angle PAO = \angle QCO$ (§ 96), $\angle AOP = \angle QOC$ (§ 69), etc.

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8. Prove both AD and $HF \parallel$ and $= BC$ by §§ 146, 101, 155, and Ax. 1.
9. Use figure, p. 81 (text-book). Then $AB + BC > AC$ (§ 78). $BC + CD > BD$ (§ 78). $CD + AD > AC$ (§ 78). $AD + AB > BD$ (§ 78). Add, and divide by 2 (Ineq. Axs. 1 and 2).
12. Construct $\angle$s of 45° at each end of the given diagonal. Use §§ 102, 115, 92.
13. Construct the $\perp$ bisector of one diagonal (§ 128), and on it from the point of intersection mark off parts $= \frac{1}{2}$ the other diagonal.

PAGE 84

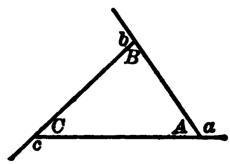
2. No. All polygons except triangles.

PAGE 86

1. 4.
2. 8.
3. 28.
4. 36.
5. 540°.
6. 1080°.
7. 1440°.
8. 2880°.
9. 120°.
10. $128\frac{4}{7}$°.
11. 144°.
12. $157\frac{1}{2}$°.
13. 181°.
14. 118°.

16. Denote the angles of the quadrilateral by a, b, c, d. Let $a + b = 180°$. Then $a + b + c + d = 360°$ (§ 167). $\therefore\ c + d = 180°$ (Ax. 3).

17. Complete the figure thus:



Then
$$A + a = 180° \ (\S\ 64)$$
$$B + b = 180° \ (\S\ 64)$$
$$C + c = 180° \ (\S\ 64)$$

$\therefore\ A + B + C + a + b + c = 540°$. (Ax. 2).

But $A + B + C = 180°$ (§ 102). $\therefore\ a + b + c = 360°$ (Ax. 3).

18. Denote the angles of the ▱ in order by a, b, c, d. If $a = 90°$, then $b = 90°$ (§ 98), $c = d = 90°$ (§ 155).

19. $\left(\dfrac{2n - 4}{n}\right) 90 = 108$, whence $180n - 360 = 108n$; $72n = 360$. $\therefore\ n = 5$. *Ans.*

20. No; yes; no.

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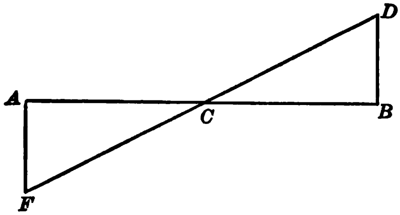
6. Divide the given perimeter into 3 equal parts by § 170.

1. (Group 16). Use § 80.
2. Use figure, p. 64 (text-book). Thus,
   Given $\angle ABC$, $BQ = BR$, BP the bisector of $\angle ABC$.
   To prove $\triangle BQP = \triangle BRP$.
   Proof. Use § 79.
3. Use § 79. Use figure of Ex. 2, p. 92.
4. § 83. Also $\triangle BAE = \triangle EDC$. For $\angle BAC = \angle DCA$ (corr. $\angle$s of $=$ ▲s). $\angle DAC = \angle BCA$ (same reason). $\therefore\ \angle BAE = \angle DCE$ (Ax. 3). $AB = DC$ (Hyp.). $\angle B = \angle D$ (corr. $\angle$s of $=$ ▲s). $\therefore\ \triangle BAE = \triangle DEC$ (§ 80).
5. Use § 80.

6.



Given ACB and FCD straight lines; $AC = CB$; AF and $BD \perp AB$.
To prove $\triangle ACF = \triangle CDB$.
Proof. $AC = CB$ (Hyp.); $\angle ACF = \angle DCB$ (§ 69); $\angle A = \angle B$ (§ 63). $\therefore$ $\triangle ACF = \angle CDB$ (§ 80).

7. Use figure, p. 78, (text-book). Thus,
Given line AC intersecting the line BD at the point F, and $AF = FC$, $BF = FD$.
To prove $\triangle BFC = \triangle AFD$, and $\triangle BFA = \triangle CFD$.
Proof. Use § 79.

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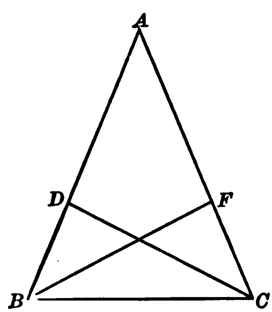
8.



Given $\triangle ABC$, $AB = AC$, and $BD = FC$.
To prove $\triangle DBC = \triangle FBC$, and $\triangle ADC = \triangle AFB$.
Proof. In ▲ DBC and FBC, $BC = BC$ (Ident.), $DB = FC$ (Hyp.), $\angle DBC = \angle FCB$ (§ 82). $\therefore$ $\triangle DBC = \triangle FBC$ (§ 79). Also in ▲ ABF and ADC, $\angle A = \angle A$ (Ident.), $AB = AC$ (Hyp.), $DB = FC$ (Hyp.). $\therefore$ $AD = AF$ (Ax. 3). $\therefore$ $\triangle ABF = \triangle ADC$ (§ 79).

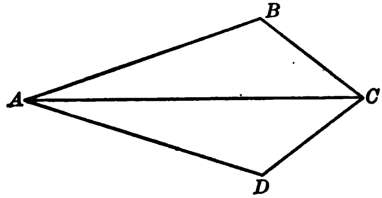
9. $AC = BD$ (Ax. 2). Use § 83.

10. The angles included by the legs are = (§ 63). $\therefore$ The ▲ are = (§ 79).

11. $AP = QC$ (Ax. 3). Use § 79.

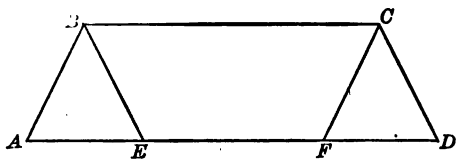
12. Two pairs. Thus, using the figure of Ex. 8,
Given the $\triangle ABC$, $AB = AC$, $CD \perp AB$, $BF \perp AC$.
To prove $\triangle DBC = \triangle BFC$, and $\triangle ABF = \triangle ADC$.
Proof. In $\triangle$ DBC and BFC, $BC = BC$ (Ident.), $\angle DBC = \angle FCB$ (§ 82). $\therefore$ $\triangle DBC = \triangle BFC$ (§ 110). Also in $\triangle$ ADC and ABF, $AB = AC$ (Hyp.), $\angle A = \angle A$ (Ident.). $\therefore$ $\triangle ADC = \triangle ABF$ (§ 110).

13.



Given the quadrilateral $ABCD$, $AB = AD$, and $\angle BAC = \angle DAC$.
To prove $\triangle BAC = \triangle DAC$.
Proof. Use § 79.

14.



Given $BC \parallel AD$, $AB = CD$. $BE \parallel CD$, $CF \parallel AB$.
To prove $\triangle ABE = \triangle CFD$.
Proof. In $\triangle$ ABE and CFD, $AB = CF$, $BE = CD$ (§ 155). $\angle ABE = \angle FCD$ (§ 112). $\therefore$ $\triangle ABE = \triangle FCD$ (§ 79).

PAGE 92

1. See Ex. 1, p. 89.
2. See Ex. 3, p. 89.
3. See Ex. 5, p. 89.
4. See Ex. 8, p. 90.

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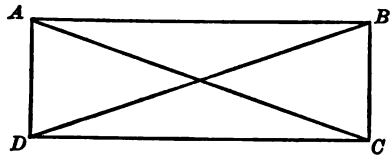
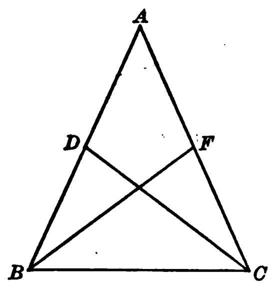
5. See Ex. 1, p. 50.
6. Use § 79.
7. See Ex. 12, p. 90.

8. 

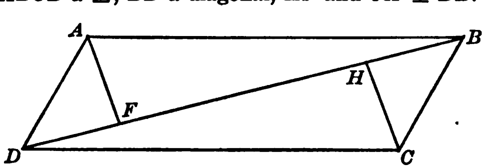
Given the rectangle $ABCD$.
To prove the diagonals AC and BD equal.
Proof. In the $\triangle\!\triangle$ ADC and BDC, $DC = DC$ (Ident.), $AD = BC$ (§ 155), $\angle ADC = \angle BCD$ (§§ 149, 63). $\therefore$ $\triangle ADC = \triangle DBC$ (§ 79). $\therefore$ $AC = DB$ (corr. sides of $=$ $\triangle\!\triangle$).

9. 

Given $\triangle ABC$, $AB = AC$, D the midpoint of AB, and F the midpoint of AC.
To prove $BF = DC$.
Proof. In the $\triangle\!\triangle$ ABF and ADC, $AB = AC$ (Hyp.). $AF = AD$ (Ax. 5.) $\angle A = \angle A$ (Ident.). $\therefore$ $\triangle ABF = \triangle ADC$ (§ 79). $\therefore$ $BF = DC$.

10. Use the figure of Ex. 8, p. 90.
Given the $\triangle ABC$, $CD \perp AB$, and $BF \perp AC$; and $CD = BF$.
To prove $AB = AC$.
Proof. In the $\triangle\!\triangle$ CDB and BFC, $BC = BC$ (Ident.), $DC = FB$ (Hyp.). $\therefore$ $\triangle DBC = \triangle FBC$ (§ 117). $\therefore$ $\angle DBC = \angle FCB$ (corr. $\angle\!\angle$ of $=$ $\triangle\!\triangle$). $\therefore$ $AB = AC$ (§ 115).

11. **Given** $ABCD$ a $\square$, BD a diagonal, AF and $CH \perp DB$.



To prove $AF = HC$.

Proof. In the $\triangle$ AFB and DHC, $AB = DC$ (§ 155), $\angle ABF = \angle CDH$ (§ 96). $\therefore \triangle AFB = \triangle DCB$ (§ 110), etc.

12. Use § 79.

13. Use §§ 82, 79.

14. Put the letter B at the vertex. Then $\angle BAP = \angle BPA$ (§ 82). $\therefore \angle OAP = \angle OPA$ (Ax. 5). $\therefore OA = OP$ (§ 115).

15. $\angle B = \angle C$ (§ 82). $\angle ADE = \angle B$ (§ 97). $\angle AED = \angle C$ (§ 97). $\therefore \angle AED = \angle ADE$ (Ax. 1). $\therefore AD = AE$ (§ 115).

16. $\triangle BAC = \triangle DAC$ (Ex. 4, p. 89). $\therefore \angle DAC = \angle BCA$ (corr. $\angle$s of $=$ $\triangle$). $\therefore AE = EC$ (§ 115).

17. Use $\triangle$ ADQ and PBC and § 79.

PAGE 94

1. See Ex. 1, p. 89.

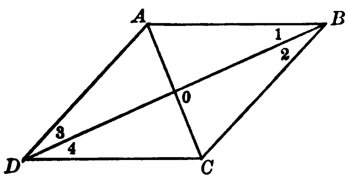
2. See Ex. 5, p. 89.

3. See Ex. 4, p. 89.

4. See Ex. 8. p. 90.

5. See Ex. 12, p. 90.

6.



Given $ABCD$ a $\square$ in which $AB = BC = CD = DA$.

To prove $\angle ADB = \angle CDB$, and $\angle ABD = \angle CBD$; also $\angle DAC = \angle BAC$, and $\angle DCA = \angle BCA$.

Proof. $\angle 1 = \angle 4$ (§ 96). But in $\triangle ABD$, $AB = AD$ (Hyp.). $\therefore \angle 1 = \angle 3$ (§ 82). $\therefore \angle 3 = \angle 4$ (Ax. 1), etc. Or, the theorem may be proved by the use of $\triangle$.

7. See Ex. 1, p. 39.

8. See Ex. 15, p. 93.

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9. $\angle DCE = \angle ECA$ (Hyp.). $\angle A = \angle ECA$ (§ 79). $\angle B = \angle DCE$ (§ 97). $\angle A = \angle B$ (Ax. 1).

10. $\angle DCE = \angle B$ (§ 97). $\angle A = \angle ECA$ (§ 96). But $\angle A = \angle B$ (Hyp.). $\therefore \angle DCE = \angle ECA$ (Ax. 1).

11. $\angle CBD = \angle BPR$ (§ 96). $\angle PBR = \angle CBD$ (Hyp.). $\therefore \angle BPR = \angle PBR$ (Ax. 1). $\therefore BR = PR$ (§ 115).

If from any point in the bisector of an angle a line is drawn parallel

to one side of the angle, and meeting the other side, the triangle thus formed is isosceles.

12. $\angle ABC = \angle ACB$ (§ 82). $\therefore$ $\angle DBC = \angle ECB$ (§ 66). $\therefore$ $\triangle DBC = \triangle BCE$ (§ 79), etc.

13. Use § 66 (complements of equal $\angle$s are equal).

14. Denote the small $\angle$s to the left of the vertex by 1, 2; those to the right by 3, 4. Then $\angle 1 = \angle 4$ (§ 69); $\angle 2 = \angle 3$ (§ 69). But $\angle 3 = \angle 4$ (Hyp.). $\therefore$ $\angle 1 = \angle 2$ (Ax. 1).

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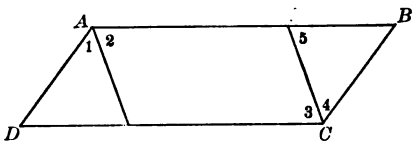
1. Use figure of Ex. 2, p. 92 (text-book.) Show $\triangle DCE = \triangle BCA$ (§79). $\therefore$ $\angle E = \angle B$ (corr. $\angle$s of $=$ $\triangle$s). $\therefore$ $DE \parallel BA$ (§ 89).

2. § 96, Ax. 5, § 89.

3. $\angle DCA = \angle A + \angle B$ (§ 103). $\therefore$ $2 \angle DCE = 2 \angle B$ (Ax. 9). $\therefore$ $\angle DCE = \angle B$ (Ax. 5). $\therefore$ $CE \parallel AB$ (§ 91).

4.



Given the $\square$ $ABCD$, AF bisects $\angle DAB$, CH bisects $\angle DCB$.
To prove $AF \parallel HC$.
Proof. $\angle DAB = \angle DCB$ (§ 155); $\angle 2 = \angle 3$ (Ax. 5). But $\angle 3 = \angle 5$ (§ 96); $\angle 2 = \angle 5$ (Ax. 1). $\therefore$ $AF \parallel CH$ (§ 91).

5. Use the figure of Ex. 8, p. 93.
Given the lines AB and DC, AD and $BC \perp DC$, $AD = BC$.
To prove $AB \parallel DC$.
Proof. $AD \parallel BC$ (§ 92), $AD = BC$ (Hyp.). $DABC$ is a $\square$ (§ 161).

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1. 57°; 85°.

2. 135°.

3. 125°; $180° - \dfrac{p° + q°}{2}$.

4. 70°; 20°; 50°.

5. 140°.

6. § 168. $\dfrac{(12 - 2)\,180°}{12}$, or 150°.

7. 5; 9; 35; $\dfrac{n(n-3)}{2}$. In general, in a polygon of n sides the number of diagonals $= {}_nC_2 - n = \dfrac{n(n-1)}{2} - n = \dfrac{n^2 - n - 2n}{2} = \dfrac{n(n-3)}{2}$. Or we may reason thus: from each vertex $n - 3$

diagonals may be drawn. $\therefore$ from n vertices $n(n-3)$ diagonals are drawn; but each of these is used twice. The number of distinct diagonals is $\frac{n(n-3)}{2}$.

8. Denote the supplementary adj. $\angle s$ by $2\angle a$ and $2\angle b$. Then $2\angle a + 2\angle b = 180°$ (§ 68). $\therefore \angle a + \angle b = 90°$ (Ax. 5).

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9. Denote the $\|$ lines by AB and CD, the transversal by PQ, the bisectors by PR and QR. Then $\angle BPQ + PQD = 2$ rt. $\angle s$. (§ 98). $\therefore \angle RPQ + \angle PQR = 1$ rt. $\angle$ (Ax. 5). . . . (1)
But $\angle R + \angle RPQ + \angle PQR = 2$ rt. $\angle s$ (§ 102) . . . (2)
Subtract (1) from (2), $\angle R = 1$ rt. $\angle$ (Ax. 3).

10. $\angle BCA = \angle A$ (§ 82), $\angle BCD = \angle D$ (§ 82). $\therefore 2\angle A + 2\angle D = 180°$ (§ 102). $\therefore \angle A + \angle D = 90°$, or $\angle BCA + \angle BCD = 90°$ (Ax. 9).

1. Let x be the complement. $\therefore 3x =$ the $\angle$. $\therefore 3x + x = 90°$. $\therefore x = 22\frac{1}{2}°$, $3x = 67\frac{1}{2}°$. *Ans.*

2. 90°; 45°. 3. 36°; 72°; 108°; 144°. 4. 30°; 60°; 90°.

5. Let $x =$ the less $\angle$. Then $x + 30° =$ the greater $\angle$. $\therefore x + x + 30° = 180°$. $\therefore x = 75°$; $x + 30° = 105°$. *Ans.*

6. $x + 2x + x + 2x = 360°$, etc.; 60°; 120°. *Ans.*

7. The vertex $\angle = 180° - 105° = 75°$. Hence, $x + 2x + 75° = 180°$, etc. 35°; 70°; 75°. *Ans.*

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8. 9. 9. 7; 12; 6.

10. Denote the number of sides by n. Then $\frac{2n-4}{n} = \frac{7}{4}$ (§ 168). $\therefore n = 16$. *Ans.*

11. The exterior $\angle s$ at the base are $180° - a$ and $180° - b$. The vertex $\angle$ of the $\triangle$ is $180° - (a + b)$ (§ 102). $180° - a + 180° - b - (180° - a - b) = 180° - a + 180° - b - 180° + a + b = 180°$.

12. The exterior $\angle$ of the small $\triangle$ on the base (to the right) is $b + x$. $\therefore a + x = b + x$ (§ 103). $\therefore a = b$ (Ax. 3).

13. In the original $\triangle$ the $\angle s$ are b, x, x. $\therefore b + x + x = 180°$ (§ 102). $\therefore \angle b = 180° - 2x$. $\therefore \frac{1}{2}b = 90° - x$ (Ax. 5). But $a = 90° - x$ (§ 106). $\therefore a = \frac{1}{2}b$ (Ax. 1).

14. $x + y = 180°$ (Ax. 5). $\therefore$ each pair of opposite sides in the figure is $\parallel$ (§ 94), etc.

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1. Use § 83.

2. Use the figure of Ex. 14, p. 90 (omitting line BE). Draw $CF \parallel AB$. Then $BA = FC$ (§ 157). But $BA = CD$ (Hyp.). $\therefore CF = CD$ (Ax. 1). $\therefore \angle CFD = \angle D$ (§ 82). But $\angle A = \angle CFD$ (§ 97). $\therefore \angle A = \angle D$ (Ax. 1).

3. If the angles which the legs of a trapezoid make with a base of the trapezoid are equal, the trapezoid is isosceles.
Draw the same auxiliary line as in the preceding Ex. and reverse the process of proof.
Exs. 2 and 3 may also be proved by drawing ⊥s from the extremities of the upper base to the lower base and using §§ 92, 157, 117, 111.

4. Draw a line through the vertex of $\angle b$ parallel with AB and use §§ 101, 96, and Ax. 2.

5. Draw the same auxiliary line as in Ex. 4.

6. Then prove $\triangle ADF = \triangle FDC$ by §§ 100, 63, 79.

7. Use the figure of Ex. 6 (text-book).
Let $\angle A = 2 \angle B$. Draw the median DC as an auxiliary line. Then $DA = DC$ (Ex. 6). $\therefore \angle DCA = \angle A$ (§ 82). Also $DB = DC$ (Ex. 6). $\therefore \angle DCB = \angle B$. Adding, $\angle A + \angle B = \angle DCA + \angle DCB = 90°$ (Ax. 2). $\therefore 3 \angle B = 90°$, or $2 \angle B = 60°$. $\therefore \angle DCA = 60°$ (Ax. 1). $\therefore \angle ADC = 60°$ (§ 102). $\therefore AC = AD$ (§ 115) $= \frac{1}{2}AB$ (Hyp.).

8. From P draw $PT \perp AD$. $\therefore PT \parallel BC$ (§ 92), $\angle APT = \angle B$ (§ 97). $\therefore \triangle ARP = \triangle ATP$ (§ 110). $\therefore PR = AT$. But $PQ = TD$ (§ 157). Use Ax. 2.

9. Denote the vertex between A and C by F, and the other vertex by H. Draw AC and BD. Then $ABDC$ is a ▱ (§ 161). $\therefore AC = BD$ (§ 155). $\therefore \triangle AFC = \triangle BHD$ (§ 83). $\therefore \angle F = \angle H$. Also $\angle ABD = \angle ACD$ (§ 155). $\angle HBD = \angle ACF$ (corr. ∠s of = △s). Adding, $\angle HBA = \angle FCD$ (Ax. 2), etc.

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10. Draw $DX \perp AC$, meeting AC in the point X, and draw $EY \perp AC$, and meeting AC produced at Y. Prove $\triangle ADX = \triangle CEY$ (§ 110). $\therefore \triangle DXF = \triangle FYE$ (§ 111), etc.

2. For if the two lines are parallel, the alt. int. ∠s are = (§ 96), which is contrary to the hypothesis. Hence the two given lines are not ||.

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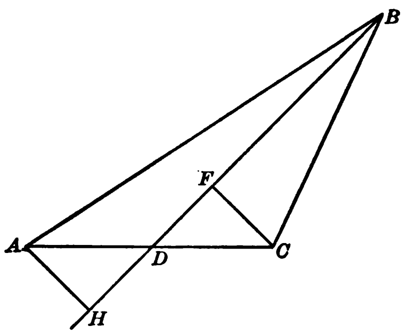
4. *CPQ* is a straight line (Constr.). ∴ ∠ *BPQ* = ∠ *CPA* (§ 69). But ∠ *BPD* = ∠ *CPA* (Hyp.) ∴ ∠ *BPQ* = ∠ *BPD* (Ax. 1). ∴ *PQ* coincides with *PD*. ∴ *PD* is in the same straight line with *CP* (for it coincides with *PQ*, which is in the same straight line with *CP*).

5. Produce the bisector of one of the vertical ∠s and show that the bisector of the other vertical ∠ coincides with the produced line (use the same method of proof as in Ex. 4).

6. Let *BE* and *DC* intersect at the point *F*. If *DC* and *BE* bisect each other show △ *BFD* = △ *FEC* (§ 79). ∴ ∠ *FBD* = ∠ *FEC* (corr. ∠s of = △s). ∴ *DB* || *EC* (§ 89), which is impossible, since *AB* and *AC* meet in the point *A*. ∴ *BE* and *DC* cannot bisect each other.

1. (Group 24). Use the figure of Ex. 13, p. 90. Let ∠ *BAC* = ∠ *DAC*, and ∠ *BCA* = ∠ *DCA*; prove △ *ABC* = △ *ADC*. Use § 80.

2.



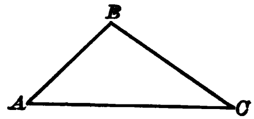
Given the △ *ABC*, the median *BH*, *AH* and *FC* ⊥ *BH*.
To prove *AH* = *FC*.
Proof. In the △s *AHD* and *DFC*, *AD* = *DC* (Hyp.). ∠ *ADH* = ∠ *FDC* (§ 69). ∠ *AHD* = ∠ *CFD* (§ 63). ∴ △ *AHD* = △ *DFC* (§ 110). ∴ *AH* = *FC* (corr. sides of = △s).

3. Use the figure of Ex. 8, p. 90. Let *ABC* be the given △, *CD* ⊥ *AB*, *BF* ⊥ *AC*; prove △ *DBC* = △ *BFC* (§ 117), etc.

4. Denote the point of intersection by F. Then $AF + FC > AC$ (§ 78), $FD + FB > DB$ (§ 78). Add, etc.

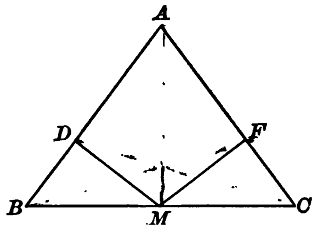
5. 68°; 34°; 102°; etc.

6.



Given AB any side of the $\triangle ABC$, and $AC > BC$.
To prove $AB > AC - BC$.
Proof. $AB + BC > AC$ (§ 78). Subtracting BC from each member of the inequality, $AB > AC - BC$ (Ineq. Ax. 1, § 133).

7.



Given $\triangle ABC$, $AB = AC$, $BM = MC$, $MD \perp AB$, $MF \perp AC$.
To prove $DM = MF$.
Proof. In the $\triangle$ BDM and MFC, $\angle B = \angle C$ (§ 82). $BM = MC$ (Hyp.). $\therefore \triangle BDM = \triangle MFC$ (§ 110). $\therefore DM = MF$, etc.

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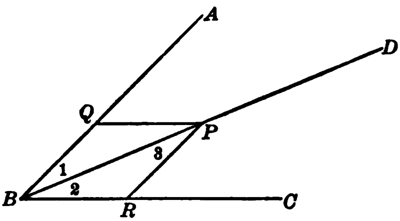
8. If the ⊥s drawn from the midpoint of one side of a $\triangle$ to the other two sides are equal, then the other two sides are equal, and the $\triangle$ is isosceles. Use the figure of Ex. 7. **Prove** $\triangle BDM = \triangle MFC$ (§ 117). $\therefore \angle B = \angle C$. $\therefore AB = AC$ (§ 115).

9. Denote the vertex $\angle$ by $2x$. Then each base $\angle = 90° - x$ (§ 102). Hence an ext. $\angle$ at the base $= 2x + 90° - x$ (§ 103), etc.

10. Draw an auxiliary line from the vertex of $\angle a$ to the vertex of $\angle d$, and produce this line through vertex of $\angle d$. Use § 103 (twice) and add (Ax. 2).

11. $AC + CD > AD$ (§ 78), or $BC + CD > AD$ (Ax. 9). Also $\angle ACD$ is 120° (§ 103). $\therefore \angle ACD > \angle D$ (§ 105). $\therefore AD > AC$ (§ 135), or $AD > AB$ (Ax. 9).

12.



Given BD the bisector of the $\angle ABC$, $PQ \parallel BC$, $PR \parallel AB$.
To prove $PQBR$ a rhombus.
Proof. $\angle 1 = \angle 2$ (Hyp.), $\angle 1 = \angle 3$ (§ 96). $\therefore \angle 2 = \angle 3$ (Ax. 1). $\therefore BR = PR$ (§ 115). But $BQ = PR$ and $QP = BR$ (§ 155). $\therefore PR = BR = BQ = QP$ (Ax. 1).

13. Use the figure of p. 37 (text-book).
Given $\triangle ABC$, $BD \perp AC$, $AD = DC$.
To prove $AB = BC$.
Proof. Show $\triangle ADB = \triangle BDC$ by § 79, etc. Or use § 120.

14. $\angle B = \angle C$ (§ 82). $\therefore 2 \angle C + 4 \angle C = 180°$ (§ 102). $\therefore 6 \angle C = 180°$. $\therefore \angle C = 30°$ (Ax. 5). $\therefore \angle BED = 60°$ (§ 102). $\therefore \angle EAF = \angle B + \angle C = 2 \angle C = 60°$ (§ 103). $\therefore \angle EFA = 60°$ (§ 102).

15. Use the figure of Ex. 6, p. 94.
Given the quadrilateral $ABCD$, in which the diagonals AC and BD intersect at rt. $\angle$s in the point O, $AO = OC$, and $BO = OD$.
To prove $ABCD$ a rhombus.
Proof. In the $\triangle$s AOB and BOC, $AO = OC$ (Hyp.), $BO = BO$ (Ident.), $\angle BOA = \angle BOC$ (§ 63). $\therefore \triangle AOB = \triangle BOC$ (§ 79). $\therefore AB = BC$ (corr. $\angle$s of $=$ $\triangle$s). In like manner it can be proved that $BC = CD = AD$. Hence $ABCD$ is a $\square$ (§ 160), and a rhombus (§ 148). If the diagonals of the given figure are equal the figure may be shown to be a square.

16. Use the figure of Ex. 8, p. 90. Let DC and BF intersect at the point O and draw OA. Prove $\triangle DBC = \triangle BFC$ (§ 110). $\therefore \angle OBC = \angle OCB$. $\therefore BO = OC$ (§ 115). Hence prove $\triangle BOA = \triangle AOC$ (§ 83), etc.

17. See figure, p. 64 (text-book). Let $PQ \perp AB$, $PR \perp BC$. Then, in the quadrilateral $PQBR$, $\angle PQB + \angle QBR + \angle BRP + \angle$

RPQ = 4 rt. $\angle$s (§ 167). But $\angle PQR + \angle PRB$ = 2 rt. $\angle$s (Hyp.) Subtract, etc.

18. $AP \parallel QC$ (Hyp. and § 146), $AB = DC$ (§ 155). $\therefore AP = QC$ (Ax. 3). $\therefore APCQ$ is a ▱ (§ 161).

19. Use figure of Ex. 8, p. 93. Prove $\triangle ADB = \triangle ACD$ (§ 83). $\therefore \angle BAD = \angle CDA$ (corr. $\angle$s of = $\triangle$s). But $\angle BAD + \angle CDA$ = 2 rt. $\angle$s (§ 98). $\therefore 2 \angle BAD$ = 2 rt. $\angle$s (Ax. 9). $\therefore \angle BAD$ = 1 rt. $\angle$. In like manner it may be shown that the other $\angle$s of $ABCD$ are rt. $\angle$s. Hence $ABCD$ is a rectangle (§ 149).

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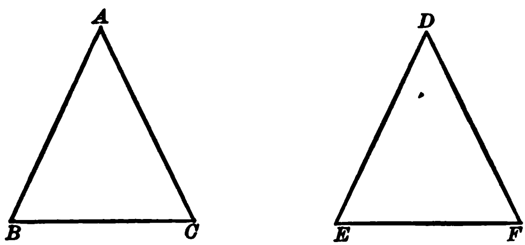
20. AC. For $\angle BCA = 60°$ (§ 64). $\therefore \angle B = 70°$ (§ 102). Use § 135.

21.



Two cases arise according as the given $\angle$ is a base $\angle$ or a vertex $\angle$.

I. Given $\triangle$s ABC and DEF, $BC = EF$, $AB = AC$, $DE = DF$, $\angle B = \angle E$.

To prove $\triangle ABC = \triangle DEF$.

Proof. $\angle C = \angle B$ (§ 82), $\angle F = \angle E$ (§ 82). But $\angle B = \angle E$ (Hyp.). $\therefore \angle C = \angle F$ (Ax. 1). Hence, prove $\triangle ABC = \triangle DEF$ (§ 80).

II. Given ABC and DEF isosceles $\triangle$s in which the base BC = the base EF and $\angle A = \angle D$.

To prove $\triangle ABC = \triangle DEF$.

Proof. Show that $\angle B = \frac{1}{2}(180° - \angle A)$ (§ 102). Also, $\angle E = (180° - D)$ (§ 102). $\therefore \angle B = \angle E$ (Ax. 1), etc. Use § 80.

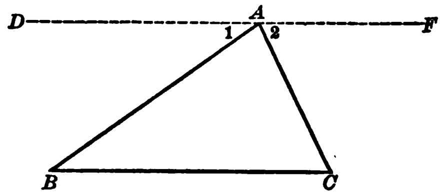
22. Use §§ 111, 102, 115, etc.

23. Produce AP to meet BC at Q.

Then $AB + BQ > AP + PQ$ (§ 78); $PQ + QC > PC$ (§ 78). $\therefore AB + BQ + PQ + QC > AP + PQ + PC$ (Ineq. Ax. 1). $\therefore AB + BQ + QC > AP + PC$ (Ineq. Ax. 1), etc.

24. $\angle APC > \angle PQC > \angle$ B (§ 87). $\therefore \angle APC > \angle B$ (§ 133, Ineq. Ax. 4).

25.



Given the $\triangle ABC$.
To prove $\angle B + \angle BAC + \angle C = 2$ rt. $\angle$s.
Proof. Draw DF through $A \parallel BC$. Then $\angle 1 + \angle BAC + \angle 2 = 2$ rt. $\angle$s (§ 68). But $\angle 1 = \angle B$, $\angle 2 = \angle C$ (§ 96). Substituting, $\angle B + \angle BAC + \angle C = 2$ rt. $\angle$s (Ax. 9).

26. $AC = BC$ (Hyp.), $RC = BQ$ (Hyp.), $AR = QC$ (Ax. 3). $\angle A = \angle C$ (§ 82). $\therefore \triangle APR = \triangle QRC$ (§ 79). $\therefore PR = QR$. In like manner show $PR = PQ$, etc.

27. Denote the point of intersection of the bisectors by R. Then show the $\triangle APR$ to be isosceles (§ 96, Ax. 1, § 115). $\therefore AP = PR$. In like manner show that $\triangle RQC$ is isosceles and $RQ = QC$. Use Ax. 2.

28. Use superposition. Or, draw a pair of corresponding diagonals and use § 79, Ax. 3, and § 79 again.

29. See Ex. 17, p. 86.

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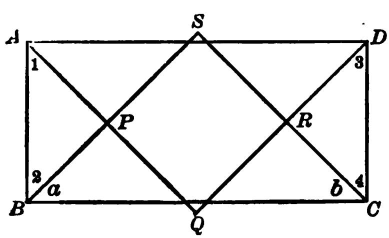
30. See Ex. 17, p. 86.
31. Use Ex. 9, p. 98, and § 92.
32. Use Ex. 31. Then show that the sides of the rectangle are equal.

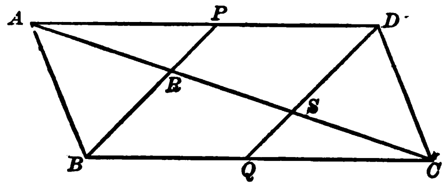


Thus, on the figure, $\angle 1 = \angle 2 = \angle 3 = \angle 4$ (each $= \frac{1}{2}$ a rt. $\angle$). $AB = CD$ (§ 155). $\therefore \triangle ABP = \triangle CRD$ (§ 119).

$\therefore$ $BP = RC$ (corr. sides of $=$ ▲). But $\angle a = \angle b$ (each $= \frac{1}{2}$ a rt. $\angle$). $\therefore$ $BS = CS$ (§ 115). $\therefore$ $PS = SR$ (Ax. 3).

33. §§ 101, 155, 83.

34.



Given $ABCD$ a ▱, $AP = PD$, $BQ = QC$.
To prove $AR = RS = SC$.
Proof. $PD = BQ$ (Ax. 5). $\therefore$ $PDQB$ is a ▱ (§ 161). In $\triangle ADS$, $AR = RS$ (§ 169). Also in $\triangle CRB$, $RS = SC$ (§ 169). $\therefore$ $AR = RS = SC$ (Ax. 1).

35. $\triangle ABP = \triangle CQD$ (§ 79). $\therefore$ $\angle BPQ = \angle DQP$ (§ 66). $\therefore$ $BP \parallel QD$ (§ 89). Use § 161. Six.

36. Use Ex. 2, p. 100, and § 98.

37. Use Ex. 2, p. 100, and § 79.

1. $22\frac{1}{2}°$; $67\frac{1}{2}°$; $146\frac{1}{4}°$; $11\frac{1}{4}°$.

2. By the law of reflection, $\angle POA' = \angle ROB'$; or, $90° - x = \angle POB - y + \angle BOB'$. $\therefore$ $90° - x = 90° - y + x$. $\therefore$ $y = 2x$.

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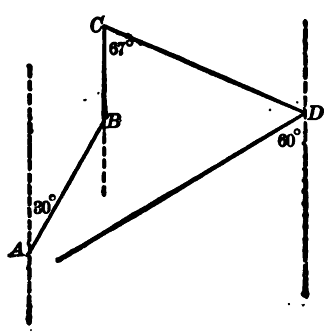
3. See Ex. 8, p. 83.

5.



7. But $a + b + c = 180°$ (§ 102). $\therefore$ $c = 90°$ (Ax. 3).

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1. §§ 57, 59.
2. §§ 15, 96, 97, 98, 100, 101, 157, 158, 169.
3. §§ 55, 56, 58, 69, 99.
4. §§ 60, 61, 63, 64, 65, 66, 67, 68, 69, 112, 113, 114, 126.
5. §§ 70, 78, 87, 102, 103, 104, 105, 134, 135.
6. §§ 79, 80, 83, 107, 109, 110, 111, 117, 138, 139.
7. §§ 142, 160, 161, 162.
8. §§ 146, 155, 156, 159, 167.
9. § 148, also see Ex. 6, p. 82.

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10. See §§ 73, 74.
11. See §§ 73, 74.
12. Make the three angles of one $\triangle$ = the corresponding angles of the other, but the corresponding sides unequal.
13. When two straight lines intersect after one of the angles of intersection has been measured, Prop. I enables us to determine the other angles without the labor of measuring them, etc.
15. § 172.
16. § 174.

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1. Prop. II enables us to determine without effort the equality of two arcs in a circle or equal circles, when the central angles which these arcs subtend are equal, etc.

PAGE 114

1. $1\frac{1}{2}$ in.

PAGE 115

1. 1 in.
2. Prop. V enables us to determine without effort the equality of two chords in a circle or equal circles, when the arcs subtended by these chords are known to be equal.
3. No.

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1. $\therefore \overset{\frown}{AB} = \overset{\frown}{CD}$ (§ 198). To each of these add $\overset{\frown}{BC}$, etc.
2. $\therefore \overset{\frown}{PR} = \overset{\frown}{QS}$ (§ 198). $\therefore \overset{\frown}{PQ} = \overset{\frown}{RS}$ (Ax. 3). $\therefore$ Chord PQ = chord RS (§ 200), etc.

3. $\overset{\frown}{AC} > \overset{\frown}{BD}$ (Hyp.). Subtract $\overset{\frown}{BC}$ from each. $\therefore\ \overset{\frown}{AB} > \overset{\frown}{CD}$ (Ineq. Ax. 1). $\therefore$ chord $AB >$ chord CD (§ 201).

4. If A, B, C, and D are four points taken in succession on a semicircle, and chord $AB >$ chord CD, then $\overset{\frown}{AC} > \overset{\frown}{BD}$.

   For $\overset{\frown}{AB} > \overset{\frown}{CD}$ (§ 201). Add $\overset{\frown}{BC}$ to each, etc.

8. 120°.

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3. Construct an arc of 45° and bisect it.
4. The point F; no.

PAGE 121

1. $\frac{1}{2}$ sec.; $\frac{2}{3}$ sec.

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1. Then $SO = OR$ (§ 207). Prove $\triangle SPO = \triangle OPR$ by § 117.
2. Use § 207.
3. No. Use § 78.
4. Use the figure of p. 121 (text-book). Chord $AB =$ chord CD (§ 207). $\therefore\ \overset{\frown}{AB} = \overset{\frown}{CD}$ (§ 198).
5. $\angle C + \angle D$ is suppl. $\angle COD$ (§ 102);
   $\angle g + \angle h$ is suppl. $\angle COD$ (§ 68);
   $\therefore\ \angle C + \angle D = \angle g + \angle h$ (§ 66). $\therefore\ 2\angle D = 2\angle h$ (Ax. 9), etc.
6. The tangents are $\perp$ the diameter (§ 210), and $\therefore$ $\parallel$ (§ 92).
7. Use § 121.

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1. 4.
2. An infinite number.
3. The circles intersect.

PAGE 129

1. Use § 78.
2. Ⓢ touching externally.
3. One $\odot$ outside the other.
4. Intersecting Ⓢ or one $\odot$ within the other.
5. Intersecting Ⓢ.
6. No.
7. Yes.
8. One circle is wholly within the other.

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1. One internal, two external.
2. One external.
3. Two external.
4. None.
5. Two internal and two external.

1. (Group 30). Use the figure, p. 127 (text-book). Draw AB. $PA = PB$ (§ 216), $OA = OB$ (§ 188), $PO \perp AB$ (§ 121).
2. Use figure 3, p. 142 (text-book). $PA = PB$ (§ 216). $\therefore \angle PAB = \angle PBA$ (§ 82). $\therefore \angle PAB = \frac{1}{2}(180° - 60°) = 60°$ (§ 102), etc.

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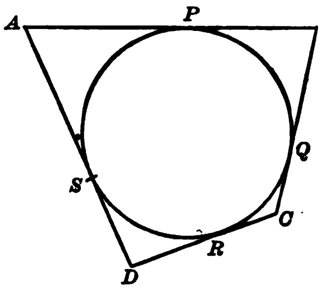
3. $RP = RA$ (§216), $PQ = QB$ (§ 216). $\therefore RP + PQ = RA + QB$ (Ax. 2).
4. 

On the figure, $\left.\begin{aligned} PA &= SA, \\ PB &= QB, \\ RD &= SD, \\ RC &= QC. \end{aligned}\right\}$ (§ 216).

Adding, $AB + CD = AD + BC$ (Ax. 2).

5. Use the method of Ex. 4.
6. Use the method of Ex. 4.
7. In the figure of Ex. 4, if $ABCD$ becomes a ▱, $AB = DC$, and $BC = AD$ (§ 155). $\therefore AB + AB = AD + AD$ (Ex. 4 and Ax. 9). $\therefore 2\,AB = 2\,AD$. $\therefore AB = AD$ (Ax. 5), etc.
8. $AP = PC$ (§ 216), $PB = PC$ (§ 216). $\therefore AP = PB$ (Ax. 1).
9. Use same method of proof as in Ex. 8.
10. $OA = OP$ (§ 188). $\therefore \angle OAP = \angle OPA$ (§ 82). Also $O'B = O'P$ (§ 188). $\therefore \angle O'BP = \angle O'PB$ (§ 82). $\therefore \angle OAP = \angle O'BP$ (Ax. 1). $\therefore OA \parallel O'B$ (§ 91).

PAGE 132

12. PC is 160 yd. (§ 216). Erect ⊥s to AB and CD at A and C. Use § 212.

PAGE 133

4. Yes.
5. No.

PAGE 135

1. $\frac{1}{3}$.

2. $\frac{60}{45}$ or $\frac{4}{3}$.

3. Take a given line as a radius, and the ends of the line as a center in turn.

4. Use each vertex of the $\triangle$ as a center and a side as the radius.

5. Use each vertex of an equilateral $\triangle$ as a center and one half the side of the $\triangle$ as a radius.

PAGE 138

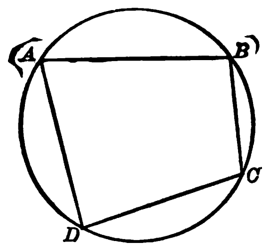
1. $46°$; $134°$.

2. $60°$; $30°$.

3. $\angle DAC = 25\frac{5}{7}°$; $\angle DAB = 12\frac{6}{7}°$.

PAGE 139

4.



On the figure, $\angle B \stackrel{m}{=} \frac{1}{2}\,\overset{\frown}{ADC}$ (§ 235). Also $\angle D \stackrel{m}{=} \frac{1}{2}\,\overset{\frown}{ABC}$ (§ 235). Adding, $\angle B + \angle D \stackrel{m}{=} \frac{1}{2}$ the circle $ABCD$ (Ax. 2). $\therefore \angle B + \angle D = 180°$, etc.

5. $360°$.

6. $\angle S = 52\frac{1}{2}°$, etc.

7. $540°$.

8. $\angle APB = 90°$ (§ 238). Prove $\triangle APB = \triangle PBC$ (§ 79).

9. $37\frac{1}{2}°$.

10. Use § 238.

11. Then at one end of the diameter construct an angle equal to the given acute angle, etc. Or at one end of the hypotenuse construct an $\angle$ = the given acute $\angle$ (§ 86). From the other end of the hypotenuse draw a $\perp$ to the other side of the acute $\angle$ (§ 129), etc.

12. On the given hypotenuse as a diameter construct a semicircle (§ 128, Post. 3). With one end of the given hypotenuse as a center and the given leg as a radius, describe an arc intersecting the semicircle. From the point of intersection draw lines to the extremities of the given hypotenuse. Use § 238.

13. Construct a circle and in it draw a chord smaller than the radius. At one end of the chord construct an angle of 120° (twice 60°), etc.

PAGE 140

1. 66°; 114°.
2. $124° = \frac{1}{2}(52° + \widehat{CB})$. $\therefore$ 196°. *Ans.*
3. 34°.

PAGE 141

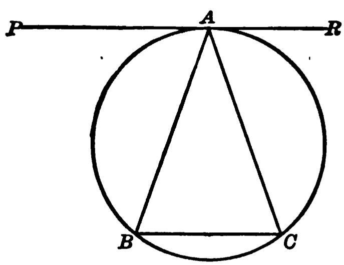
1. 47°.
2. $\angle CPD = 40°$; $\angle CPA = 50°$, etc.
3. $\widehat{PDC} - (360° - \widehat{PDC}) = 128°$. $\therefore$ $\widehat{PDC} = 244°$, etc.

PAGE 143

1. $\angle P = 37°$, etc.
2. Denote the $\widehat{AFC}$ by x. Then $80° = \frac{1}{2}(360° - x - x)$. $\therefore$ $x =$ 100°.

PAGE 144

1. §§ 200, 202, 207, 216.
2. (1) §§ 196, 237, 241, 242; (2) See Ex. 4, p. 139; (3) §§ 210, 238.
3. §§ 195, 198, 202, 235, 244.
4. Use figure 3, p. 142 (text-book). $\angle PAB \stackrel{m}{=} \frac{1}{2}\widehat{AFB}$ (§ 241). Also $\angle PBA \stackrel{m}{=} \frac{1}{2}\widehat{AFB}$. $\therefore$ $\angle PAB = \angle PBA$ (Ax. 1).
5.



Let ABC be an inscribed $\triangle$ in which $AB = AC$, and let PR be tangent to the $\odot$ at A. Then $\widehat{AB} = \widehat{AC}$ (§ 198). $\angle PAB = \angle RAC$ (measured by = arcs, § 241). But $\angle ABC \stackrel{m}{=}$

$\frac{1}{2}\overset{\frown}{AC}$ (§ 235). $\therefore\ \angle ABC = \angle RAC$ (Ax. 1). $\therefore\ PR \parallel BC$ (§ 89). The converse is true.

6. On the figure, p. 140 (text-book), if $\angle DPB$ should be made a rt. $\angle$, $\angle DPB \overset{m}{=} \frac{1}{2}(\overset{\frown}{DB} + \overset{\frown}{AC})$ (§ 240). $\therefore\ \frac{1}{2}(\overset{\frown}{DB} + \overset{\frown}{AC}) = 90°$. $\therefore\ \overset{\frown}{DB} + \overset{\frown}{AC} = 180°$ (Ax. 4).

7. $\angle BAC \overset{m}{=} \frac{1}{2}\overset{\frown}{AB}$ (§ 241). But $\angle AOB \overset{m}{=} \overset{\frown}{AB}$. $\therefore\ \frac{1}{2}\angle AOB \overset{m}{=} \frac{1}{2}\overset{\frown}{AB}$ (Ax. 5). $\therefore\ \angle BAC = \frac{1}{2}\angle AOB$ (Ax. 1).

8. On the figure to Ex. 4, p. 139, produce DC through C to F. Then $\angle BCF$ is supplement of $\angle BCD$ (§ 32). But $\angle DAB$ is supplement of $\angle BCD$ (Ex. 4, p. 139). $\therefore\ \angle A = \angle BCF$ (§ 66).

9. Use § 238.

10. Use the figure of Ex. 18, p. 145, as given in the Key (omitting the circumscribed rhombus). Then chord $AD \parallel$ chord BC. $\overset{\frown}{AB} = \overset{\frown}{DC}$ (§ 244). In like manner $\overset{\frown}{BC} = \overset{\frown}{AD}$. Adding, $\overset{\frown}{ABC} = \overset{\frown}{ADC}$. $\therefore\ \overset{\frown}{ABC}$ = semicircle. $\therefore\ \angle B$ = rt. $\angle$ (§ 238). In like manner $\angle$s A, C, D are rt. $\angle$s, etc.

11. Use § 235.

12. Then $\overset{\frown}{ABC} = \overset{\frown}{BCD} = \overset{\frown}{CDE}$, etc. (Ax. 4). $\therefore\ \angle ABC = \angle BCD = \angle CDE$, etc. (§ 237).

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13. §§ 97, 230, 235.

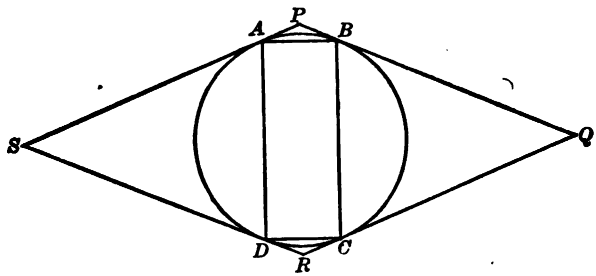
14. $\angle B = \angle D$ (each $\overset{m}{=} \frac{1}{2}\overset{\frown}{AC}$, § 235). Use § 107.

15. $\overset{\frown}{AD} = \overset{\frown}{BE}$ (§ 244). $\therefore\ \angle CAD = \angle AED$ (§§ 241, 235, Ax. 1). Also, $\angle C \overset{m}{=} \frac{1}{2}(\overset{\frown}{ABE} - \overset{\frown}{AD})$, or $\frac{1}{2}\overset{\frown}{AB}$ (§ 242). $\therefore\ \angle C = \angle AEB$ (§ 235, Ax. 1). Use § 107.

16. Draw $\perp$s from the centers of the circles to the cutting line. These $\perp$s are $\parallel$ (§ 92), and $\therefore$ = (§ 157). Then use § 207.

17. $\angle BAE = \angle BCD$ (§ 114). $\therefore\ \overset{\frown}{BD} = \overset{\frown}{BE}$, since they measure equal $\angle$s.

18.

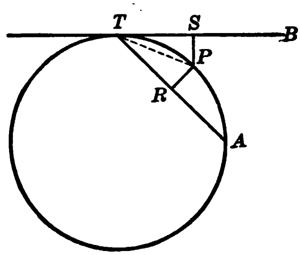


In the figure $AB \parallel CD$ (Hyp.). $\therefore \overset{\frown}{BC} = \overset{\frown}{AD}$ (§ 244). $\therefore \angle SAD = \angle SDA = \angle QBC = \angle QCB$ (§ 241). Chord BC = chord AD (§ 155). $\therefore \triangle SAD = \triangle QBC$ (§ 80). $\therefore BQ = SA$ (corr. sides of = $\triangle$s).

But $AP = BP$ (§ 216). Adding, $SP = PQ$ (Ax. 2). In like manner, $PQ = QR = RS$, etc. Use § 160.

1. Draw OB. Then $\angle AOB = \angle BOC$ (§ 196). $\therefore \triangle AOB = \triangle BOC$ (§ 110), etc.

2.



On the figure, $\overset{\frown}{PT} = \overset{\frown}{AP}$ (Hyp.). Draw PT. Then $\angle ATP = \angle PTS$ ($\overset{m}{=}$ $\frac{1}{2}$ of equal arcs AP and TP, §§ 235, 241). $\therefore \triangle RPT = \triangle STP$ (§ 110), etc.

3. Produce OD to meet the circle at F. Then $\overset{\frown}{AF} = \frac{1}{2} \overset{\frown}{AC}$ (§ 202). $\therefore \angle ABC \overset{m}{=} \overset{\frown}{AF}$ (§ 235). But $\angle AOF \overset{m}{=} \overset{\frown}{AF}$ (§ 230), etc.

4. §§ 110, 207. The converse is true.

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5. Draw a line from the center to the point of intersection. Use §§ 204, 117.

6. Use the figure, p. 127 (text-book) and draw AB. Then $\angle APO$ = complement of $\angle AOP$ (§§ 210, 102). Also $AB \perp OP$ (Ex. 1, Group 30, p. 130). $\therefore \angle OAB$ is complement $\angle AOP$ (§ 102). $\therefore \angle OAB = \angle APO$ (§ 66). But $\angle APB = 2 \angle APO$, etc.

7. Draw the auxiliary lines as indicated in the figure. Prove the two ⊿ formed, equal by § 198, Ax. 3, §§ 200, 237, 80.

8. Draw the chord AC. Then $\overset{\frown}{AB} = \overset{\frown}{CD}$ (§ 198). $\overset{\frown}{BD} = \overset{\frown}{BD}$ (Ident.). Adding, $\overset{\frown}{ABD} = \overset{\frown}{CDB}$ (Ax. 2). $\therefore \angle A = \angle C$ (measured by equal arcs, § 235). $\therefore PA = AC$ (§ 115).

9. On the diagram, p. 127 (text-book), prove $\angle POA = \angle POB$. Denote the equal ⦞ at the center of the diagram for Ex. 9 by a, a; b, b; c, c; d, d. Then $2a + 2b + 2c + 2d = 360°$ (§ 67). $\therefore a + b + c + d = 180°$ (Ax. 5), etc.

10. $\angle APM = 60°$ (§ 235, for $\overset{\frown}{AC} = 120°$). $\therefore \triangle PAM$ is equilateral (§ 102). $\therefore PA = AM$. But $AB = AC$ (Hyp.). $\therefore \angle PAB = \angle MAC$ (each $60° - \angle BAM$). Use § 80, etc.

11. Show that the radii, given and drawn, form two pairs of equal ⦞, and denote these ⦞ by a, a and b, b. Then $\angle a + \angle b = 90°$ (Hyp.). $\therefore 2 \angle a + 2 \angle b = 180°$. Hence, two of these radii form a straight line. (§ 65). Use §§ 210, 92.

12. Draw a $\perp$ from the common center to AD. Use § 202 and Ax. 3.

13. $\angle PAB = \angle PBA$ (§ 82). $\therefore \angle x = \angle y$ (§ 66). $\therefore \overset{\frown}{BDC} = \overset{\frown}{ACD}$ (§ 235). $\therefore \overset{\frown}{AC} = \overset{\frown}{BD}$ (Ax. 3). Use §§ 200, 207, etc.

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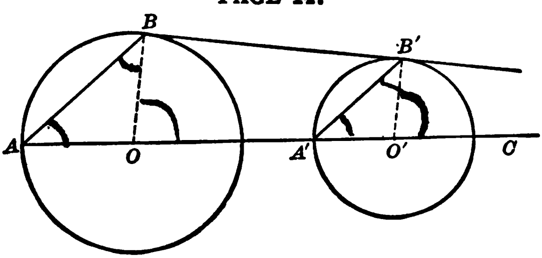
14.



Draw radii OB and $O'B'$ from the centers O and O' to the point of tangency B and B'.

$\therefore OB$ and $O'B'$ are $\perp BB'$ (§ 210). $\therefore OB \parallel O'B'$ (§ 92). $\angle BOA' = \angle B'O'C$ (§ 97). But $\angle A = \angle ABO$ (§ 82). $\therefore \angle BOA' = 2 \angle A$ (§103). Similarly, $\angle B'O'C = 2 \angle B'A'O'$. $\therefore 2 \angle A = 2 \angle B'A'O'$ (Ax. 1). $\therefore \angle A = \angle B'A'O'$ (Ax. 5). $\therefore AB \parallel A'B'$ (§ 97).

15. §§ 238, 202.

16. Denote the tangent by RS. $\angle D = \angle SPB = \angle APR = \angle C$ (§§ 235, 241, 69, Ax. 1). Use § 89.

17. Draw the common chord. Use Ex. 4, p. 139, §§ 64, 66, 94.

18. The circle will pass through both the center of the square and the vertex of the rt. $\triangle$ (§ 238 used conversely). Use §§ 198, 235.

19. Through P draw a common tangent intersecting AB in R. Then (Ex. 8, p. 131), $RA = RP = RB$. $\therefore$ describe a $\odot$ with R as center and RA as radius. Use § 238.

20. Use § 238 conversely and § 237.

21. Draw the auxiliary line PQ. Then $\angle PQA = 1$ rt. $\angle = \angle PQB$ (§ 238). $\therefore AQB$ is a straight line (§ 65).

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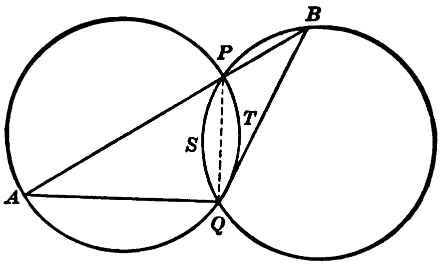
22.



Draw the common chord PQ. Then $\overset{\frown}{PSQ} = \overset{\frown}{PTQ}$ (§ 198). $\therefore \angle B = \angle A$ (measured by $\frac{1}{2}$ equal arcs, PSQ and PTQ, § 235). $\therefore QA = QB$ (§ 115).

23. The number of angular degrees in the inscribed $\angle ABC$ is one half the number of degrees of arc in $\overset{\frown}{AC}$.

1. The diameter through the given point is greatest chord (§ 194). The chord $\perp$ this diameter is least. For draw any other chord through the point and a $\perp$ to this chord from the center. Use §§ 136, 209.

2. $PO - CO < PC$ (Ex. 6, group 24, p. 102). $\therefore$ $PA < PC$ (Ax. 9). Also $PO + OD > PD$ (§ 78). $PB > PD$ (Ax. 9).

3. $OC < OP + PC$ (§ 78), $OP + PB < OP + PC$ (Ax. 9). $\therefore$ $PB < PC$ (Ineq. Ax. 1). Also $OD + OP > PD$ (§ 78). $\therefore$ $AP > PD$ (Ax. 9).

4. From O' draw a line $\parallel$ CD to meet OR in T. Then $O'T \perp OR$ (§ 100). $\therefore$ $OO' > O'T$ (§ 135). But $O'T = RS$ (§ 157). $\therefore$ $OO' > RS$ (Ax. 9). $\therefore$ $AB > CD$ (§ 202, Ax. 2, Ineq. Ax. 2).

5. This circle will pass through the point B (§ 238 used conversely). Let this $\odot$ cut OP at point R. $\therefore$ $\angle OAB = \angle ORB$ (§ 237). But $\angle ORB > \angle P$ (§ 87), etc.

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1. For if it is not, on the chord opposite the rt. $\angle$, as a diameter, describe a semicircle. Either this semicircle will cut the sides of the given $\angle$, or one of the sides produced cuts the semicircle. In either case use § 87.

2. If they are not diameters, the $\angle$s of the rectangles are not right $\angle$s (§ 239), etc.

3. If $\overset{\frown}{AB}$ is not $>$ $\overset{\frown}{DF}$, it must $=$ $\overset{\frown}{DF}$, or be less than $\overset{\frown}{DF}$, etc. (See method of proof in § 135.)

4. For the longer $\perp$ must be $\perp$ the other chord (§ 100) and bisect it (§ 202). $\therefore$ it coincides with the other $\perp$ (§ 93), etc.

5. Use § 87 and Ex. 4, p. 139.

1. (Group 36). $\angle P$ is measured by $\frac{1}{2}$ $\overset{\frown}{BC}$ (constant). $\therefore$ $\angle P$ is constant. But $\angle A$ is constant by hyp. $\therefore$ $\angle P + \angle A$ is constant.

2. $RQ = AR + QB$. To each of these equals add $TR + TQ$. $\therefore$ $RQ + TR + TQ = TA + TB$ (a constant sum).

3. See Ex. 9, p. 146. $\angle AOR = \angle POR$. $\angle BOQ = \angle POQ$. Adding, $\angle AOR + \angle BOQ = \angle POR + \angle POQ = \angle ROQ$. $\therefore$ $\angle ROQ = \frac{1}{2}$ $\angle AOB$. But $\angle AOB$ is constant. $\therefore$ $\angle ROQ$ is constant.

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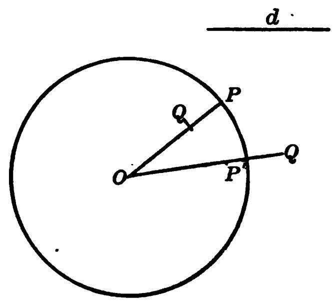
4. $\angle CBD = \angle P + \angle PCB$ (§ 103). $\angle P$ is measured by $\frac{1}{2}$ $\overset{\frown}{AB}$ (in its circle). $\angle PCB$ is measured by $\frac{1}{2}$ $\overset{\frown}{AB}$ (in its circle), etc.

5. CD is constant (Hyp.). $OP = \frac{1}{2}$ CD (Ex. 6, p. 100). $\therefore$ OP is constant.

6. In order to give a convenient formal proof of this and the remaining exs. in this group it is best to give the following
LEMMA. The locus of a point moving so as to be always at a given distance from a given fixed point is a circle whose center is the given point and whose radius is the given distance.



Given the point O, the line (segment) d, and the circle PP' whose center is O and radius $= d$.
To prove PP' the locus of the point moving so as always to be at distance d from O.
Proof. Let P be any point on the given circle. Then $PO = d$ (§ 188). Hence, every point on the given circle is at the given distance from O. Again, let Q be any point not on the given circle. Draw OQ. Then either OQ intersects the given circle at some point P', or on being produced OQ meets it at P. In the first case $OQ > OP'$, *i.e.*, $OQ > d$ (Ax. 8). In the second case $OQ < OP$; that is, $OQ < d$ (Ax. 8). Hence, every point not on the given circle is not at the distance d from O. Hence, the circle PP' is the required locus (§ 123).
To prove Ex. 6, the moving point is at the distance $r + a$ or $r - a$ from the center of the given circle (Hyp.). Hence, its locus is the two circles whose common center is the center of the given circle and whose radii are $r + a$ and $r - a$ respectively (Lemma). In case $a > r$ the second part of the answer disappears.

7. The moving point is always at a distance $\frac{1}{2}r$ from the center of the given circle (Hyp.); hence, its locus is the circle whose center is the center of the given circle and whose radius $= \frac{1}{2}$ radius of given circle (Lemma of Ex. 6).

8. The midpoints of all the given chords are equidistant from the center of the given circle (§§ 203, 207). Hence the locus of the moving point is a circle whose center, etc. (Lemma of Ex. 6).

9. Draw a line from the vertex of one of the right $\triangle$ to the mid-point of the given hypotenuse. The line thus drawn $= \frac{1}{2}$ the hypotenuse (Ex. 6, p. 100). Hence, the locus of the vertex of the given right $\angle$ is the circle having the given hypotenuse for is its diameter (Lemma of Ex. 6).

10. Ex. 10 reduces to Ex. 9 (§ 203). Hence the locus is the circle having for its diameter that radius of the original circle which is drawn to the given point on the circle.

11. The circle whose center is O and radius $= \frac{1}{2}CD$ (see Ex. 6, p. 100 and Lemma of Ex. 6).

12. Let O be the center of the given circle. On OB lay off $OC = QP$. Draw OQ amd CP. Then C is a fixed point, since QP is a constant. Also $OQPC$ is a $\square$ (§ 161); in every position of P, $CP = OQ =$ radius of given circle (§ 155). Hence the locus of P is a circle having C for its center and OQ for its radius (Lemma of Ex. 6).

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1. See figure of Ex. 2, p. 145. Let TB be a tangent, TA a chord, $\angle ATP = \angle PTB$, and prove $\overset{\frown}{TP} = \overset{\frown}{PA}$. Use §§ 235 and 241.

2. §§ 244, 200.

3. $\overset{\frown}{AC} = 35°$; $\angle T = 100°$; $\angle TAB = \angle TBA = 40°$; $\angle TBC = 57\frac{1}{2}°$. $\angle BAD = 47\frac{1}{2}° = \angle BCD$; $\angle S = 30°$; $\angle ABC = \angle ADC = 17\frac{1}{2}°$. If AD and BC intersect at O, $\angle BOD = \angle AOC = 65°$. $\angle BOA = \angle COD = 115°$.

4. By § 246, the $\parallel$ tangents bisect the circle at their points of tangency. Hence, the line joining the points of contact is a diameter (§ 192).

5. Use Ex. 7, p. 131.

6. $OB = BP$ (Hyp.). $\therefore \angle ABO = 2 \angle P$ (§§ 103, 82, Ax. 9). But $\angle A = \angle OBA$ (§ 82). $\angle AOC = \angle A + \angle P$ (§ 103) $= 2 \angle P + \angle P = 3 \angle P$.

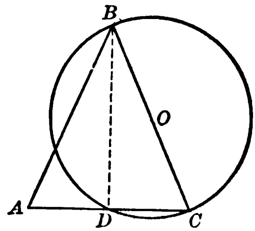
7. By § 198 each side of the square subtends an arc of 90°. $\therefore$ required $\angle = 45°$ (§ 241).

8. Draw $\perp$s from the center upon the given secants. The two rt. $\triangle$ thus formed are $=$ (§ 110). $\therefore$ chords are $=$ (§ 207). Add equal semichords to equal corresponding sides of $\triangle$.

9. Use Ex. 5, p. 149, §§ 64, 167.

10. Draw the line of centers. Prove that this passes through the point of contact of the two circles. Then use §§ 69, 82, Ax. 1, § 89.

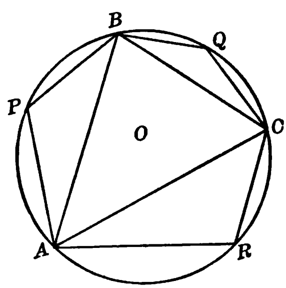
11.



Let ABC be a triangle in which $AB = BC$ and O a circle having BC for its diameter and intersecting AC in D. Then BDC is a right $\angle$ (§ 238). $\therefore \triangle ABD = \triangle DBC$ (§ 117). $\therefore AD = DC$.

12. Draw a line from one end of the chord to the midpoint of the arc. Use §§ 235, 241, Ax. 1, § 89.

13.



Given the circle O with the inscribed $\triangle ABC$, P any point on $\overset{\frown}{AB}$, Q on $\overset{\frown}{BC}$, R on $\overset{\frown}{AC}$.

To prove $\angle APB + \angle BQC + \angle ARC = 4$ rt. $\angle$s.

Proof. $\angle APB \overset{m}{=} \frac{1}{2}(\overset{\frown}{BQC} + \overset{\frown}{ARC})$ (§ 235). $\angle ARC \overset{m}{=} \frac{1}{2}(\overset{\frown}{BQC} + \overset{\frown}{APB})$ (§ 235). $\angle BQC \overset{m}{=} \frac{1}{2}(\overset{\frown}{ARC} + \overset{\frown}{APB})$ (§ 235). Adding, $\angle APB + \angle ARC + \angle BQC \overset{m}{=} \frac{1}{2}(2\overset{\frown}{BQC} + 2\overset{\frown}{ARC} + 2\overset{\frown}{APB})$, or by $\overset{\frown}{BQC} + \overset{\frown}{ARC} + \overset{\frown}{APB}$; that is, by the circle. Hence, the sum of the $\angle$s named $= 4$ rt. $\angle$s.

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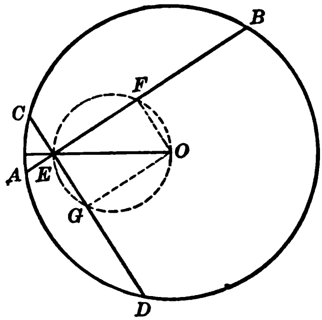
14. Sum = 6 rt. $\measuredangle$. Method of proof is the same as that used in Ex. 13.

15. The $\perp$ bisector of the line joining the points (§§ 188, 122).

16. Given circle O, chord $AB >$ chord CD, and E their point of intersection.

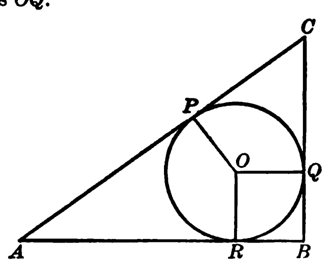


To prove $\angle OED > \angle OEB$.

Proof. On OE as a diameter describe a circle intersecting AB at F and CG at G. Draw OF and OG. Then $\measuredangle$ OFE and OGE are rt. $\measuredangle$ (§ 238). $OG > OF$ (§ 208). $\overset{\frown}{OG} > \overset{\frown}{OF}$ (§ 201). $\angle OEG > \angle OEF$ (§ 235).

Converse. Of two unequal chords in a circle, the chord which makes the less angle with the diameter through their point of intersection is the greater. The converse is true.

17. Given rt. $\triangle ABC$ with rt. $\angle$ at B with the inscribed circle O whose radius is OQ.

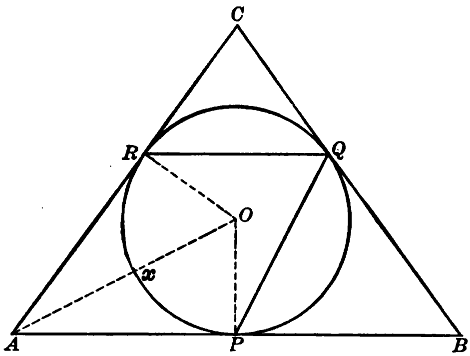


To prove that leg AB + leg $BC = AC + 2\,OQ$.

Proof. Let P, Q, R, be the points of contact, and draw OP,

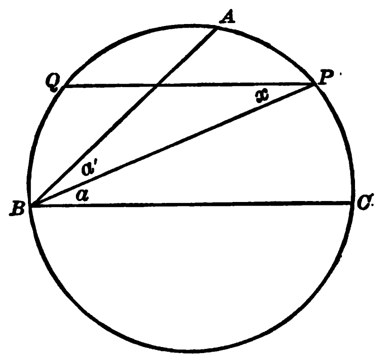
OR, OQ. $AR = AP$ (§ 216), $CQ = CP$ (§ 216) (1). △ OQB and ORB are rt. △ (§ 210), $\angle B$ = rt. $\angle$ (Hyp.). $\therefore OR \parallel QB$ and $RB \parallel OQ$ (§ 92). $\therefore QB = OR$ and $RB = OQ$ (§ 155) (2). Adding (1) and (2), $AB + BC = AC + 2\,OQ$ (Ax. 2).

18.



Draw also line OA intersecting the circle at x. Then $\angle POx = \angle ROx$ (§ 216, and corr. △ in = △). $\therefore \widehat{Px} = \widehat{Rx}$ (§ 195). $\angle POx \stackrel{m}{=} \widehat{Px}$ (§ 230). $\angle PQR \stackrel{m}{=} \frac{1}{2}\,\widehat{RP}$ or $\widehat{Px}$ (§ 235). $\therefore \angle PQR = \angle POx$ (Ax. 1). But $\angle POA$ is complement of $\angle PAO$ (§ 102). $\angle PAO = \frac{1}{2}\,\angle RAP$ (§ 216). $\therefore \angle PQR$ is complement of $\frac{1}{2}\,\angle RAP$ (Ax. 9).

19.



Given PB the bisector of the inscribed $\angle ABC$. PQ a chord $\parallel BC$.

To prove $AB = PQ$.

Proof. $\angle x = \angle a$ (§ 96). $\angle a = \angle a'$ (Hyp.). $\therefore \angle x = \angle a'$ (Ax. 1). $\overset{\frown}{BQ} = \overset{\frown}{AP}$ (§ 235). Add AQ to each arc. $\therefore \overset{\frown}{AQB} = \overset{\frown}{QAP}$ (Ax. 2). $\therefore$ chord AB = chord PQ (§ 200).

20. Draw OC and AC. Then, in the $\triangle OCP$, CA = radius (Ex. 6, p. 100). $\therefore \triangle OCA$ is equilateral. But $\angle CAO = \angle P + \angle PCA$ (§ 103), or $60° = 2 \angle P$ (§ 82, Ax. 9). $\therefore 30° = \angle P$. $\therefore \angle AEP = 60°$ (§ 106). Similarly by use of $\triangle BOC$, $\angle B = 30°$. $\therefore \angle CDE = 60°$, etc.

21. Draw the radius to the point of contact. Use § 169 and prove that the rt. $\triangle$, in which the lines to be proved equal are the hypotenuses, are equal (§ 79).

1. Construct any two chords of the circular rim of the fragment and then construct the $\perp$ bisectors of these chords. The point of intersection of the $\perp$s is the center (§ 204).

2. See diagram. Use § 238.

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3. Use § 238.

4. $\angle TPC = \angle PAC + \angle PCA$ (§ 103). But $PA = PC$ (§ 216), etc.

5. Prove $\angle y = \angle O$, and $\angle z = \angle FO'C$.

6. In $\triangle FEH$, $z = y + 180° - x$ (§ 103). $\therefore x - y + z = 180°$. *Ans.*

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7. Use §§ 210, 114.

8. $AP \parallel EP'$, owing to great distance of the sun. $\therefore AP$ (if produced) $\perp QE$ (§ 100). Use § 114. $41° 30'$. *Ans.*

9. Latitude of the place is $\overset{\frown}{QZ}$. Hence, if the sun is north of the celestial equator, $\overset{\frown}{QZ} = \overset{\frown}{QS_1} + \overset{\frown}{S_1Z}$. If south, $\overset{\frown}{QZ} = \overset{\frown}{SZ} - \overset{\frown}{QS}$.

10. At the north pole the latitude = 90°. Hence, QS_1 = sun's elevation above the horizon, which must equal the sun's declination.

11. $6° 7'$. At any hour of the day.

12. From the diagram, $\angle HOS = x + x = 2x$.

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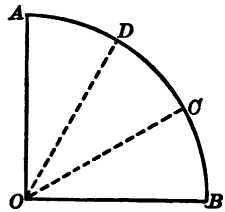
3. $112\frac{1}{2}° = 90° + 22\frac{1}{2}°$.



Let AOB be the given rt. ∠, and $\widehat{AB}$ a quadrant with O as a center. With OA as a radius and A and B as centers, describe arcs intersecting AB in D and C. Draw OD and OC. ∠ AOB will be trisected by OD and OC. For ∠ $DOB = 60°$ (angle of an equilateral △ if chord DB is drawn). ∴ ∠ $AOD = 30°$, etc.

5. No.

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4. (1) In diagram on p. 159 make ∠ $A = 60°$, $m = 2$ in., $n = 1\frac{3}{4}$ in.
(2) On diagram p. 159, make ∠ $A = 60°$, $m = 2$ in., $n = 1$ in.

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3. Use § 238. Hence, since BC is a vertical line AC is ⊥ to it and ∴ horizontal.

PAGE 161

3. Construct an equilateral triangle.

PAGE 162

1. Bisect the exterior angles of the triangle.
2. In first diagram, let A, B, C, D be the vertices of the square, and O the midpoint of AB. A ⊥ from O to AC is the radius of the semicircle.

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2. 81°; 81°; 99°; 99°.

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1. From the center of the given circle draw a ⊥ the given line (§ 129). At the points where this ⊥ intersects the given circle draw tangents to the circle (§ 262). Use §§ 210, 92.

2. From the center of the given circle draw a line || the given line (§ 95). At the points where the line drawn meets the given circle, draw tangents to the circle (§ 262). Use §§ 210, 100.

3. Draw the chord connecting the two given points. Draw the radius ⊥ this chord (§ 129). Through the two given points draw lines || this radius. Use §§ 202, 100, 155, 207.

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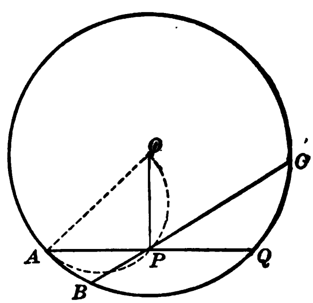
4. At any point in the given line draw a line which makes an angle with the given line equal to the given ∠ (§ 86). Through the given point draw a line || last line drawn (§ 95).

6. Take any point on one of the given || lines as a center, and with a radius equal to the given line-segment describe an arc intersecting the other || line. Connect this point of intersection with the center of the arc. Through the given point draw a line || last line drawn. Use § 157. (Two solutions in general.)

7. Let O be the center of the given circle. With any point P on the given circle as a center and the given line as a radius describe an arc intersecting the given circle at Q. Draw the chord PQ. With O as a center and the distance of PQ from O as a radius describe a circle. From A draw a tangent to this last circle (§ 264). This tangent produced is the chord required. Use §§ 188, 207. (Two solutions in general.)

8. Draw a concentric circle with the given distance as radius. Use § 264, etc.

9.



Let O be the given circle, A the given point, and BC the given chord. It is required to draw a chord from A to be bisected by BC.

Draw the radius OA. On OA as a diameter describe the semicircle intersecting BC at P (§ 128). Draw AP and produce it to meet the given circle at Q. Then AQ is the chord required. For $\angle APO$ is a rt. $\angle$ (§ 238). $\therefore AP = PQ$ (§ 202).
There is no solution when the circle drawn (APO) does not intersect the given chord (BC).

1. (Group 40.) That is, draw the $\perp$ bisector of the line CD (§ 128). Use § 120.
2. Draw the $\perp$ bisector of CD. See Ex. 1. The $\perp$ bisector will meet the circle in two, one, or no points, according to circumstances
3. Draw the bisectors of the $\measuredangle$ made by the given intersecting lines (§ 84). Use § 125. (Two solutions in general.)
4. With the given point as a center and d as a radius describe an arc intersecting the given line (Post. 3). (Two solutions in general.)

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5. Construct a circle with radius a, and another with radius b, having A and B for their respective centers. In general the two circles will intersect in two points and there will be two solutions. When $a + b =$ line AB, the circles touch and there is one solution. When $a + b < AB$, the circles do not meet and there is no solution.
6. That is, draw the $\perp$ bisector of the line connecting the two given points (§ 128). Also draw two lines $\parallel$ the given line, at the given distance from it (§§ 85, 95). (Two solutions.)
7. § 128, Post. 3.
8. §§ 128, 84.
9. A line $\perp$ the given line at the given point (§ 210).
10. The moving point is always at the distance r from the given point (§ 188). Hence, the locus is the circle having the given fixed point for its center and r for its radius. (See Ex. 6, p. 150, Lemma.)
11. The moving point is equidistant from the two given lines (§§ 210, 188). Hence, the locus is the bisectors of the $\measuredangle$ made by the given intersecting lines (§ 127).
12. The moving point is equidistant from the given $\parallel$ lines (§§ 210, 188). Hence, the locus is a line $\parallel$ the given lines and midway between them.
13. The moving point is always at the distance r from the given line (§§ 210, 188). Hence, the locus is two lines $\parallel$ the given line and at the distance r from it.

14. See Ex. 6, p. 150.

1. Construct any equilateral △. Bisect one of its ∠s (§ 84). Cut off a part of the bisector from the vertex, equal to the given altitude. At the end of the altitude remote from the vertex of the ∠, erect a ⊥ (§ 85), etc.

2. Bisect the base (§ 128). Erect a ⊥ at midpoint of the base = given altitude (§ 85), etc.

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3. At each end of the base and on the same side of the base construct an ∠ = the given ∠ (§ 86).

4. Bisect the given vertex ∠ (§ 84), make the bisector = given altitude, etc.

5. At one end of the given leg construct an ∠ = given acute ∠ (§ 86). At the other end of the leg erect a ⊥ (§ 85), etc.

6. Construct the complement of the given acute ∠, and use Ex. 5.

7. Then with the other end of the altitude as a center, describe arcs with the given sides as radii, etc.

8. Draw a line ‖ one of the given sides and at a distance = the given altitude (§§ 85, 95). With one end of the side thus used as a center, describe an arc with the other side as a radius, etc.

9. Bisect the given ∠ (§ 84) and from the vertex mark off on the bisector a part = the given diagonal. At the other end of the diagonal draw lines ‖ the sides of the given ∠ (§ 95). Use §§ 96, 115, etc.

10. At the end of the given altitude construct an ∠ = complement of the given ∠. At the other end of the altitude erect a ⊥ to the altitude, etc.

11. Bisect the given diagonals and construct a △ with the semi-diagonals as sides and the included ∠ = the given ∠ (§ 253), etc.

12. Use §§ 253, 252.

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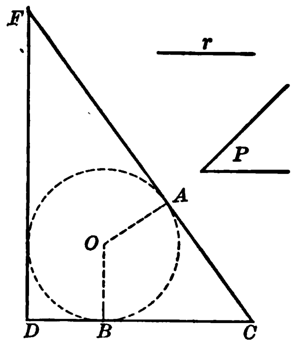
1. Draw the circumscribed ⊙. In this insert a chord = the given base. Construct the ⊥ bisector of the base (§ 128), etc.

2. Draw the circumscribed ⊙. Insert a chord in this circle = given leg. Draw the diameter from one end of this chord, etc. Use § 238.

3. Draw the circumscribed ⊙ and a diameter in this ⊙. At one end of this diameter construct an ∠ = the given acute ∠ (§ 86), etc. Use § 238.

4.

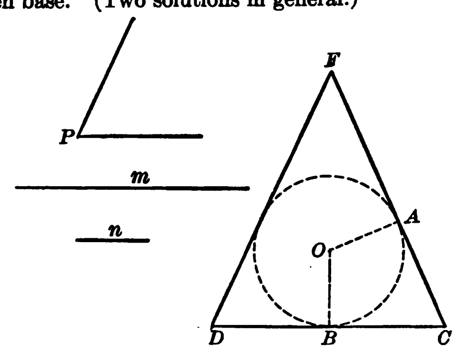


Then, at the ends of the radii of this central ∠ (A and B) draw tangents to the given circle, meeting at C.
Draw DF tangent circle and $\parallel$ OB (Ex. 1, p. 165) meeting CA produced at F and CB produced at D. Then $\triangle$ DFC is the $\triangle$ required. For $OB \perp DC$ (§ 210). $\therefore$ $FD \perp DC$ (§ 100), etc.

5. Also draw a line $\parallel$ given base at a distance = given altitude (§§ 85, 95). From the points where this $\parallel$ line intersects the circle draw lines to the extremities of the base. (Two solutions in general.)

6. On the given base construct a segment containing the given ∠ (§ 265). With the midpoint of the base as a center and with a radius = given median describe an arc intersecting the circle. From the points of intersection draw lines to the extremities of the given base. (Two solutions in general.)

7.



Let m be the given side, P the given ∠, and n the radius of the inscribed ⊙.

Draw the inscribed ⊙. At O construct $\angle AOB$ supplement of P. At A and B draw tangents to the circle O intersecting at C (§ 262). On CB produced mark off $CD = m$. From D draw DF tangent to circle O (§ 264) and meeting AC produced at F, etc.

8. Draw the circumscribed ⊙. In this circle insert a chord = the given side. At one end of this chord construct an $\angle$ = the given $\angle$, etc.

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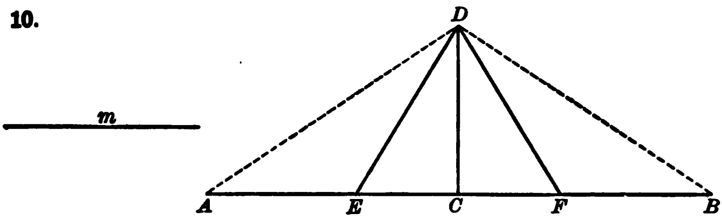
10.

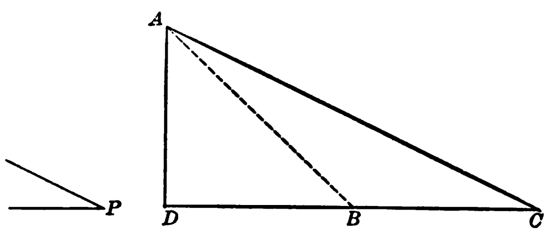


Let m be the given altitude, AB be the given perimeter. At C, its midpoint, construct $CD \perp AB$ and equal to the given altitude m. Connect AD and BD. Construct $\angle ADE = \angle A$, and $\angle FDB = \angle B$. Then EDF is △ required. For $AE = ED$, $FB = DF$ (§ 115), etc.

11. A base $\angle = \frac{1}{2}(180° - \text{vertex } \angle)$. Then follow method of Ex. 9.

12. Also at B construct an $\angle$ = given acute $\angle$, etc.

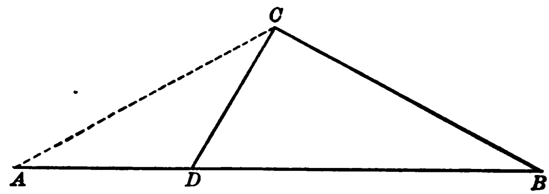
13.



Let BC be the difference of the legs and P the given acute $\angle$. Produce BC through B and construct $\angle DBA = 45°$. At C construct $\angle ACB$ = given acute $\angle P$. From A, the point where AC and AB intersect, drop $\perp AB$ to DC (§ 129). Then ADC is the △ required. For $\angle DAB = 45°$ (§ 106). $\therefore AD = DB$ (§ 115), etc.

14. Take a line equal to the sum of the legs and at one end of this line construct an $\angle$ of 45°. With the other end as a center, and the given hypotenuse as a radius, describe an arc cutting the indefinite side of $\angle$ 45°. From the point of intersection thus formed drop a $\perp$ to the line which is the sum of the legs, etc.

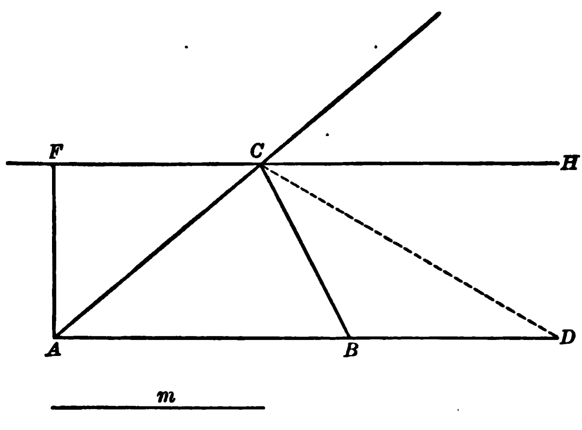
15.



Let x be the given acute $\angle$ and AB the sum of hypot. and one leg. At B construct $\angle B = \angle x$. At A construct $\angle A = \frac{1}{2}(90° - x)$. At C, the intersection of CA and CB, construct $\angle ACD = \angle A$. Then $\triangle DCB$ is the $\triangle$ required. For $\angle CDB = 2 \angle A = 90° - x$ (§ 103), etc.

16. At an end of the line = sum of two sides construct an angle = $\frac{1}{2}$ the given $\angle$. With the other end as a center and the given side as a radius describe an arc, etc.

17.



Let AD be the sum of the two sides; $\angle CAB$ = given $\angle$, m = alt. on AB. Then at A draw $AF \perp AD$ and $= m$ (§ 85).

Through F draw $FH \parallel AD$ and intersecting AC in C (§ 95). Draw CD. At C construct $\angle BCD = \angle D$. Then $\triangle ACB$ is the $\triangle$ required, etc.

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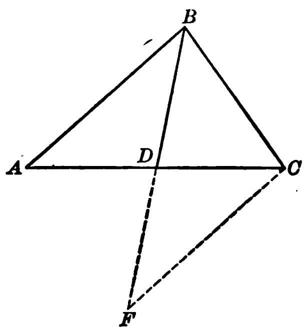
1. Construct a right $\triangle$ having the given median for its hypotenuse and the given altitude for a leg (Ex. 12, p. 139). Then with that vertex of this $\triangle$ which is opposite given altitude, as a center, and the median as a radius, describe a circle (use Ex. 6, p. 100), etc.

3. Ex. 12, p. 170.   **4.** § 252.

5.



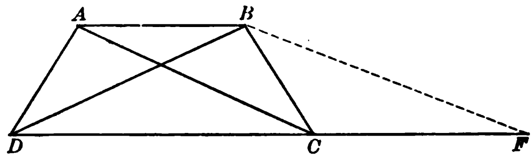
Analysis. In the $\triangle ABC$ let AB and BC and the median BD be given. Produce BD to F, making $DF = DB$. Draw FC. Then $\triangle ABD = \triangle FDC$. $\therefore FC = AB$. In $\triangle BCF$ the three sides are known. Hence,

Construction. Construct a triangle (BCF), one of whose sides (BF) is twice given median and whose other sides are the two given sides (§ 252). Draw the median CD and produce it to A, making $AD = DC$. Draw AB. Then $\triangle ABC$ is the required $\triangle$. For $\triangle ADB = \triangle FDC$ (§ 79). $\therefore AB = CF$, etc.

6. Ex. 3, p. 168. The base of the isosceles $\triangle$ = difference of the given trapezoid bases, etc.

7. Ex. 12, p. 139. The leg of the right $\triangle = \frac{1}{2}$ the difference of the given trapezoid bases, added to the shorter base of the trapezoid, etc.

8. § 252.

9. 

ANALYSIS. Let AB and CD be the bases of the trapezoid $ABCD$. Produce DC to F, making $CF = AB$. Draw BF. Then $ABFC$ is a ▱ (§ 161). $\therefore$ $BF = AC$. Hence,

CONSTRUCTION. Construct the △ (DBF) whose base (DF) = sum of bases of required trapezoid, and whose other two sides are its diagonals (§ 252). Through the vertex B draw a line (BA) ∥ base and equal to one base (CF) (§ 95). Then draw $AC \parallel BF$ and meeting DF in C. Draw BC. Then $ABCD$ is the trapezoid required. For $AC = BF$ (§ 157).

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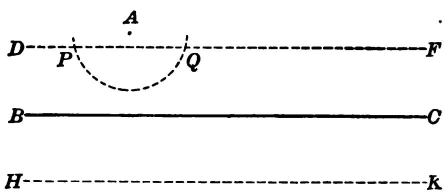


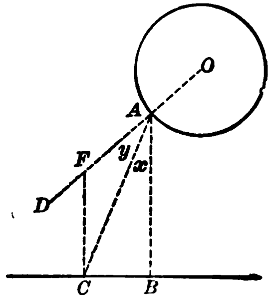
Let A be the given point, BC the given line, r the given radius. Then construct the locus of all points at the distance r from BC, viz., the lines DF and HK ∥ BC. Also construct the locus of all points at the distance r from the point A, viz., $\widehat{PQ}$ (Ex. 6, p. 150, Lemma). These loci may intersect in certain points as P and Q. Then with P and Q as centers describe circles with r as a radius. These circles will be the circles required.

2. Draw lines ∥ the second line at the distance r, etc.
3. Find a point at distance r from each point. (See Ex. 5, p. 167.)
4. Draw a line ∥ the given lines and midway between them. Find a point in this last line at a distance from the given point equal to ½ the distance between the given ∥ lines.
5. Find a point in the given line equidistant from the two given points. (See Ex. 1, p. 166.)

6. Bisect the $\angle$s made by the intersection of these lines. Use § 126 in the proof.

7. Also draw the $\perp$ bisector of AB.

8.



Draw the radius OA and produce it to D. Drop a $\perp$ AB from A to the given line BC (§ 129). Bisect the $\angle$ DAB by the line AC (§ 84), meeting BC at C. At C construct $CF \perp BC$ and meeting OD at F. Then F is the center and FC the radius of the required circle. For $\angle x = \angle y$ (Constr.), $\angle x = \angle FCA$ (§ 96). $\therefore$ $\angle y = \angle FCA$ (Ax. 1). $\therefore$ $FA = AC$ (§ 115), etc.

9. Denote the center of the given $\odot$ by O. Draw a line $CF \parallel AB$ at a distance = radius of given circle and on the opposite side of AB from the $\odot$. Through A draw a line $AC \perp$ line CF. Draw OC. At O in OC construct the $\angle COP = \angle ACO$ (§ 86). Let OP meet AC produced at P and intersect the circle at D. Then P is the center of the $\odot$ required and PA is its radius. For in $\triangle PCO$, $\angle PCO = \angle O$ (Constr.). $\therefore$ $PC = PO$ (§ 115). But $AC = DO$ (Constr.). $\therefore$ $PA = PD$ (Ax. 3), etc.

1. (Group 45). Bisect the $\angle$s of intersection of the two given lines (§ 84). From the given point draw $\perp$s to the bisectors of these $\angle$s (§ 129).

2. See Ex. 12, p. 139.

3. Bisect the $\angle$s made by the given intersecting lines (§ 84) and produce the bisectors to meet the given circle. Use § 127.

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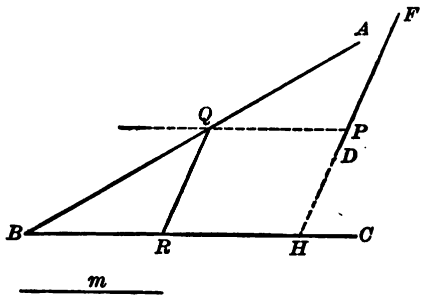
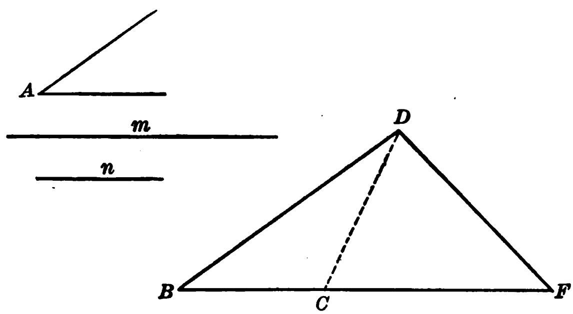
4. Bisect the $\angle$s made by the given intersecting lines (§ 84). At the given point erect a $\perp$ to the line in which the point is and produce it to meet the bisectors, etc.

5.



Let ABC be the given $\angle$, m the given line-segment, and DF last given line. Produce DF to meet BC in H. On HF mark off $HP = m$. Through P draw $PQ \parallel BC$ and meeting BA in Q (§ 95); from Q draw $QR \parallel FH$ and meeting BC at R. Then QR is the line required, etc.

6.



Let m be the given side, A the adjacent $\angle$, and n the difference of the other two sides. Then construct $\triangle BCD$ having $BC = n$, $BD = m$, $\angle B = \angle A$ (§ 253). Produce BC to F and at D in the line CD construct $\angle CDF = \angle DCF$ (§ 86). Produce DF to meet CF at F. Then $\angle BDF$ is the $\triangle$ required. For $CF = DF$ (§ 115). $\therefore$ $BF - DF = BF - CF = BC$.

7. Describe a circle with the given point as a center and the given distance as a radius.

8. Reduce to § 255.

9. Then through the extremity of the last line draw a line $\parallel$ one side of the given $\angle$ and meeting the other side at the point O.

From O draw a line through the given point to meet the other side of the given $\angle$. The last line drawn is the line required. Use §§ 69, 96, 80.

10. Put the segments of the base together in line to form the base. On this base construct a segment of a circle which shall contain the given vertex $\angle$ (§ 265). At the point of the base which separates the given segments erect a $\perp$ (§ 85) and produce it to meet the arc of the segment, etc.

11. A base $\angle = \frac{1}{2}(180° - \text{vertex } \angle)$.

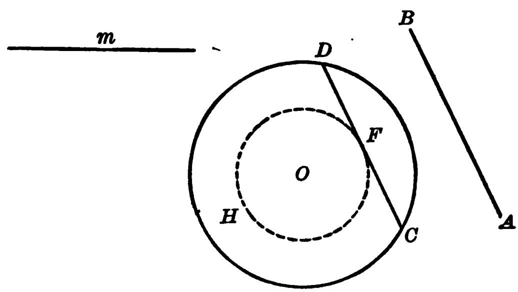
12. From the center of the given circle draw a radius to the given point on its circle. Produce this radius till the produced part = the radius of the required circle. The extremity of the produced part is the center of the required circle.

13. Upon the given hypotenuse as a diameter describe a semicircle. Draw a line $\|$ the hypotenuse at a distance = given altitude, etc. Use § 238.

14. Then, with each end of the given base as a center, and one of the given altitudes as a radius, describe an arc cutting the circle. Through each point of intersection and the nearest extremity of the base draw a straight line and produce these lines to meet. Use § 238.

15. At one end of the given altitude construct an $\angle$ = the complement of one of the given $\measuredangle$. At the same point but on the opposite side of the altitude, construct an $\angle$ = complement of the other given angle. At the other end of the altitude construct a $\perp$ to the altitude and produce it to meet the sides of the $\measuredangle$ so formed.

16.



Let O be the given $\odot$, m the given line-segment, and AB the given line. Denote the radius of the given circle by R. Bisect

the line m. Construct a rt. △ having R for its hypotenuse and $\frac{1}{2}m$ for a leg (see Ex. 12, p. 139). With O as a center, and a radius = the other leg of this rt. △, describe the circle FH. Draw $DC \parallel FH$ and tangent to circle O (Ex. 1, p. 165). Then CD is the chord required. Use §§ 210, 202, 117.

17. Construct the given ∠ and its bisector. Draw a line ∥ one side of the ∠, at a distance from it = given altitude and intersecting the other side of the ∠. Draw a line from this point of intersection through the end of the bisector, etc.

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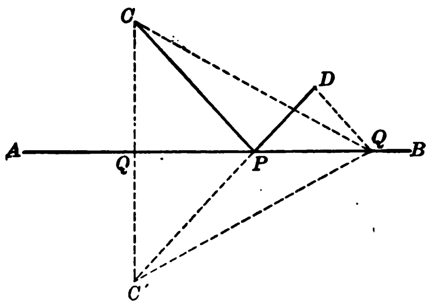
18. The locus will be a circle having as its diameter the line connecting the given point and the given center.

19.



Draw $CQ \perp AB$ and produce CQ to C', making $QC' = CQ$. Draw $C'D$ cutting AB in P. Draw CP. Then P is the point required.

Proof. In ⊿s CPQ and $C'PQ$, $CQ = QC'$ (Constr.), $QP = QP$ (Ident.), $\triangle CQP = \triangle C'QP$ (§ 79). $\therefore \angle CPQ = \angle C'PQ$. But $\angle C'PQ = \angle DPB$ (§ 69). $\therefore \angle CPQ = \angle DPB$ (Ax. 1).

20. Use the same figure and construction as in Ex. 19. Then take any other point in AB except P as Q, and draw DQ and $C'Q$ and CQ. Then in $\triangle C'DQ$, $C'P + PD < C'Q + DQ$ (§ 78). For $C'P$ substitute its equal CP. $\therefore CP + PD < C'Q + DQ$ (Ax. 9). For $C'Q$ substitute its equal CQ. $\therefore CP + PD < CQ + DQ$ (Ax. 9).

21. Using the center of the larger ⊙ as a center and the difference of the radii of the given circles as a radius, describe a circle. Use §§ 264, 210, 161, 98, 155, 211.

22. Same construction as in Ex. 21, except that the new radius = sum of given radii.

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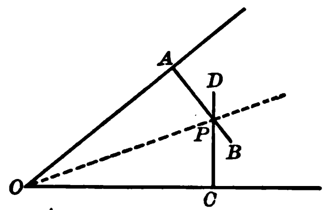
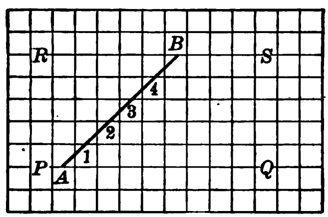
1. Locate two diameters of the circle by use of § 238.

2.



Let O be the given angle. Place the carpenter's square in the two positions, OAB and OCD, OA being equal to OC. Then OP bisects $\angle O$ (§ 117).

3.



On the squared paper let the cross line RS be 5 spaces from the || cross line PQ. Take A, any point on PQ, and with A as a center and a radius = $1\frac{1}{4}$ in., describe an arc cutting RS at B. Denote the points where AB cuts the cross lines || PQ, by 1, 2, 3, 4. Then AB is divided into 5 equal parts at 1, 2, 3, 4 (§ 169). The construction can be made on paper ruled in only one direction.

5. At D construct $\angle BDF = 60°$. Make $DF = BF$. Produce DF to H and construct $\angle KFH = 60°$. Then FK is an extension of AB. To prove this draw BF. In $\triangle BFD$ show $\angle FBD = \angle BFD = 60°$. $\therefore$ BF is an extension of AB and also of KF.

6. Use §§ 96, 80.

7. See Ex. 12, p. 132.

PAGE 176

9. In $\triangle EAN$, $\angle EAN$ is comp. of $\angle GAE$; $\angle AEN$ is comp. of $\angle EAC$. But $\angle GAE = \angle EAC$ (Constr.). $\therefore$ $\angle EAN = \angle AEN$ (§ 66). $\therefore$ a circle may be described with N as a

center and radius AN. Similarly show that H is equidistant from E and C. Through E draw $PQ \perp EH$. Then EP is tangent to arcs AE and EC. $\therefore$ AEC is a compound curve (§ 211). Etc.

10. $KG = KA$ (§ 120). $\therefore$ arc from A passes through G. Also in $\triangle EDG$, $\angle EDG = \angle HAB$, $\angle EGD = \angle HGK$. $\therefore$ $\angle EDG = \angle EGD$ (Ax. 1). $\therefore$ an arc drawn with E as a center and EG as a radius passes through D. Through G draw a line $\perp EK$. This line will be a common tangent to the arcs NG and GD (§ 211). Hence, these arcs form a compound curve.

PAGE 177

1. §§ 183, 196, 197, 200, 201.
2. §§ 184, 194, 198, 201, 202, 207, 208, 209.
3. §§ 182, 192, 202.
4. §§ 195, 197, 229, 230.
5. §§ 233, 234, 235, 237, 238, 239.
6. §§ 185, 210, 211, 212, 213, 216, 218, 242.
7. Three; two; four; three if one is at the vertex, otherwise four.
8. One; one; three.
9. See §§ 258 and 261.
10. §§ 238–239.
11. If we know that in the same circle or in equal circles two chords are equal, Prop. IX enables us to determine, without effort, that the chords are equidistant from the center, and conversely.
12. If we know that a straight line is tangent to a circle, Prop. XII enables us to determine without effort that it is $\perp$ the radius drawn to the point of contact.
13. If we know the length of one of the tangents drawn from a given point to a given circle, we are enabled to determine the length of the other tangent thus drawn, without measuring it.
14. If in a circle or in two equal circles we know the relative size of two arcs intercepted by two central angles, Prop. XVI enables us to determine without effort the relative size of these angles.

18. See § 251.

19. The number of degrees of angle in $\angle ABC$ equals the number of degrees of arc in $\overset{\frown}{AC}$.

PAGE 180

1. $5 : x = x : a.$
2. 10.
3. 15.
4. $\frac{pq}{a}.$
5. 2.4
6. .54.
7. $12\,b.$
8. 96.
9. 9; $\frac{1}{4}$.
10. $3\sqrt{2}$; .08.
11. $8\frac{1}{3}$; $\frac{1}{18}$.
12. $9\,p^2.$
13. $\sqrt{a^2 - b^2}.$

PAGE 181

1. $a : p = q : b$; $b : p = q : a$, etc.
2. $x : 3 = 2 : x + 1$; $3 : x = x : 5.$
3. $x : y = 4 : 3.$

4 $\frac{7}{8}$. 5. $\frac{b}{a}$. 6. $\frac{p+q}{a+b}$. 7. $\frac{b-d}{a-c}$.

PAGE 182

1. $x : b = a : c.$
2. $a : x = c : b.$
3. $c : a = b : x.$
4. $c : a = b : x.$
   $2c : a = b : x.$
5. $1 : x = x : 3.$
   $a + b : x = x : a - b.$

1. (§ 286). $15 : 3 = 10 : 2.$
2. $2x : 5 = 3x : 7.$
3. $\sqrt{x+3} : 5 = \sqrt{2x-1} : 7.$
4. It simplifies the proportion by the cancellation of like terms.

PAGE 183

1. $9 : 3 = 6 : 2.$
2. $2x : 5 = 3x : 7.$
3. $7x : \sqrt{3x-5} = 4x : \sqrt{2x+1}.$
4. It simplifies the proportion by the cancellation of like terms.
6. Use § 287. $3x : 2y = 8 : 2.$

PAGE 184

1. $6x : 4y = 12 : 8.$
2. $2\sqrt{x+7} : 2\sqrt{5} = 4x : 2.$
3. See Ex. 4, p. 183.

1. (§ 289). $24 : 6 = 4 : 1.$
2. $\frac{x+6}{y+6} = \frac{x+1}{y+3}.$
3. $\frac{x}{y} = \frac{2}{5}.$

PAGE 185

1. $\frac{6}{7}$.
2. $\frac{2}{5}$.

1. (§ 291). 8 : 27.
2. 8 : 125.
3. 4 : 49.
4. $\frac{9}{16}$.
5. $\frac{15}{2}$.
6. (1) When we know the value of the ratio of like roots of two quantities, § 291 enables us to find the ratio of the numbers themselves.
(2) It enables us often to find the value of an unknown quantity involved in a proportion containing radicals.

PAGE 186

1. $6x$.
2. $\dfrac{(a-b)^2}{a+b}$.
3. 12.
4. Use § 278.
5. $2x : 2 = 12 : 2$; $x = 6$.
6. $bqx = bpy$ (§ 278).
$\therefore \quad qx = py$ (Ax. 5).
$\therefore \quad x : y = p : q$ (§ 281).
7. $P : W = l : L$.
8. 40 lb.
9. Any three of the four quantities P, W, l, L being given, it enables us to find the remaining one.

11. $\dfrac{2x^2}{4x^2} = \dfrac{6 + 2\sqrt{1+x}}{8 - 4\sqrt{1-x}}$ (§ 284); hence, $\dfrac{1}{1} = \dfrac{3 + \sqrt{1+x}}{2 - \sqrt{1-x}}$ (§ 227).
12. $\dfrac{2x^3 - 6x^2}{10x - 14} = \dfrac{4x^3 + 8x^2}{-6x + 8}$ (§ 288); hence, $\dfrac{x-3}{5x-7} = \dfrac{2x+4}{-3x+4}$ (§§ 284, 227).
13. $\dfrac{x+2}{x-2} = \dfrac{3x+5}{3x-8}$ (§ 291); $\dfrac{x}{2} = \dfrac{6x-3}{13}$ (§§ 288, 227).

PAGE 188

1. 9; 6.
2. 5.
3. $\dfrac{ac}{b}$.
4. $4\frac{1}{5}$; $5\frac{1}{7}$; $2\frac{2}{35}$.

PAGE 189

3. Use the method of § 297, making $p = n$.
4. $\therefore \ a + b : a - b = c : x$ (§ 288). Use § 297.

PAGE 190

3. In general use the method of § 298, but on AP lay off the three lines m, n, p in succession.

PAGE 191

1. Yes. 2. No. 3. Yes.

PAGE 192

1. 8.
2. 1.5.
3. 6; 8.
4. 14; 7.
5. 5; 15.
6. 4; 6.

PAGE 194

1. Yes.
2. No, for $\frac{4}{10}$ does not equal $\frac{6}{12}$.
3. 32.9; 252; 23.1; 16.8.

PAGE 195

1. Yes, when they are mutually equiangular; no.
3. Use § 304. Also let BD and CA intersect in the point O. Then the △ BOA and COD are similar.
4. 8 into 6 and 2; 10 into $7\frac{1}{2}$ and $2\frac{1}{2}$.
6. § 83; § 89 (twice).

PAGE 196

1. No.
2. Because it is not given that ∠ A and A' are equal.
3. Yes.

PAGE 197

1. Use §§ 238, 305.
2. Use §§ 96, 304.
3. △ APB, FHR, PQR are similar (§§ 108, 305, 306).
4. § 305.
5. The △ are ABQ, CPQ, APD. Use § 305.
6. Draw DB and AC on the figure, p. 208 (text-book). Two pairs.
7. See Ex. 14, p. 145.
8. Use §§ 300, 97, 304, 302.
9. 372 yd. It enables us to determine the distance from a given place to an inaccessible object.

PAGE 198

10. Use §§ 97, 305.

1. 5.6; 7.

PAGE 199

1. Similarly $\frac{AH}{AC} = \frac{1}{3}$. Use Ax. 1 and § 310.

2. Let ABC and $A'B'C'$ be the given similar △ and BD and $B'D'$ the corresponding medians. Then in the △ ABD and $A'B'D'$, $\angle A = \angle A'$ (§ 302). Also $AB : A'B' = AC : A'C'$ (§ 302). But $AC : A'C' = AD : A'D'$ (§ 227). $\therefore AB : A'B' = AD : A'D'$ (Ax. 1). ∴ the △ ABD and $A'B'D'$ are similar (§ 310). In like manner the △ BDC and $B'D'C'$ may be proved similar.

3. Use §§ 97, 305.

4. In △ ABP and $A'B'P'$, $A'B' \parallel AB$ (§ 300). Prove △ ABP and $A'B'P$ similar by § 305. $\therefore AB : A'B' = BP : B'P$. Similarly, $BC : B'C' = BP : B'P$. $\therefore AB : A'B' = BC : B'C'$, etc. Use § 309.

5. In Ex. 4, change "within" to "without." Use the same method of proof as in Ex. 4.

6. Let P fall on the side AC. Prove $AB \parallel A'B'$; $BC \parallel B'C'$. Also, $\frac{AP}{A'P} = \frac{AB}{A'B'} = \frac{BP}{B'P} = \frac{BC}{B'C'} = \frac{PC}{PC'}$. $\therefore \frac{AP}{A'P} = \frac{PC}{PC'}$, whence $\frac{AP}{PC} = \frac{A'P}{PC'}$ (§ 284). $\therefore \frac{AP + PC}{PC} = \frac{A'P + PC'}{PC'}$ (§ 286). Or, $\frac{AC}{PC} = \frac{A'C'}{PC'}$. Hence, $\frac{AC}{A'C'} = \frac{PC}{PC'} = \frac{BC}{B'C'}$. Use § 309.

7. 110 ft.

8. 1085 ft.

9. By § 301, $8 + 10 : BC = 9 : 15$. $\therefore BC = 30$. *Ans.*

10. Let ABC and $A'B'C'$ be the two similar triangles. Let $AB = BC$. Then $AB : BC = A'B' : B'C'$ (§ 302). $\therefore A'B' = B'C'$ (§§ 284, 282).

PAGE 203

2. Lengths of lines drawn are $1\frac{1}{2}$ in.; $2\frac{1}{2}$ in.; 6 in.

3. $2\frac{1}{2}$ in. by $1\frac{1}{4}$ in.

4. 100 ft.

5. Facilitates (1) a more ready comparison of parts; (2) discussion of changes; (3) sending information to a distance.

PAGE 204

1. 140 rd.

2. Twice.

PAGE 205

1. Divide the squares into triangles and use §§ 310, 313.

2. $137\frac{1}{2}$ miles.

3. Distance $= 1{,}000{,}000 \times \frac{11}{4}$ in. $= 43.40 +$ mi. *Ans.*

4. Draw a pair of corresponding diagonals denoted by h and h'. Then, by § 312, $h : h' = a : a'$. $\therefore\ P : P' = h : h'$ (Ax. 1).

5. See Ex. 1, p. 199; 680 yd.

6. Prove $\frac{PA}{PA'} = \frac{AB}{A'B'} = \frac{PB}{PB'} = \frac{BC}{B'C'} = \frac{PC}{PC'}$. $\therefore\ \frac{PA}{PA'} = \frac{PC}{PC'}$ (Ax. 1). Use § 299.

7. $\frac{OB}{OA} = \frac{11}{3} = \frac{OD}{OC}$ (Hyp.). Use § 310.

8. 10.

9. 15; $7\frac{1}{2}$.

10. Prove $\triangle AKM = \triangle AMC$ (§ 80). $\therefore\ KM = MC$. $\therefore\ DM \parallel BK$ (§ 300). $AK = AC$ (corr. sides of $=$ ▲). $\therefore\ 7 - BK = 4.6$. $\therefore\ BK = 2.4$, $MD = 1.2$. *Ans.*

PAGE 207

1. 6; $2\sqrt{13}$; $3\sqrt{13}$.

2. .9; .7; $\sqrt{.63} = \frac{3}{10}\sqrt{7} = .793 +$.

3. First find $FC = .32$; then $BC = \frac{4}{25}\sqrt{5} = .357 +$.

PAGE 208

2. Construct the mean proportional between 1 and 2; between 2 and 3.

3. Construct the mean proportional between 1 in. and 5 in.

PAGE 209

1. 6.

2. 12, 6.

3. $x(11 - x) = 6 \times 4$. $\therefore$ 8, 3. *Ans.*

4. $\frac{ab}{c}$.

5. $AF = \frac{r}{2} \pm \sqrt{\frac{r^2}{4} - pq}$. The $\pm$ sign means that two constructions of the figure are possible according as AF is greater or less than FB.

PAGE 210

1. 6.

2. $4\overline{AF}^2 = 144$. $\therefore AC = 24$. *Ans.*

3. .16.

4. Let OA cut the circle in D. Then $OD = 21 - 15 = 6$. $\therefore AC \times AF = 36 \times 6 = 216$ (§ 325).

1. (Group 51.) ▲ ADC and BFC are rt. ▲ having the acute $\angle C$ in common. Use § 305.

2. ▲ AOF and ACD are rt. ▲ having the acute $\angle CAD$ in common. Use §§ 305, 306.

3. Denote a base $\angle$ of each by A. Then the vertex $\angle$ of each $\triangle = 180° - 2\angle A$. Hence the ▲ are mutually equiangular. Use § 304.

4. Use figure Ex. 10, p. 212 (text-book). Denote the point where AC and BD intersect by F. Then ▲ AFD and BFC are mutually equiangular (§ 96), and $\therefore$ similar (§ 304).

5. $\angle P$ is measured by $\frac{1}{2}$ arc AC (§ 235). Also $\angle FAC$ by $\frac{1}{2}$ arc BC (§ 235). $\therefore \angle P = \angle FAC$. Also $\angle C = \angle C$ (Ident.). $\therefore$ ▲ APC and AFC are similar (§ 305).

6. $\angle AEB$ is a rt. $\angle$ (§ 238). $\angle ABD$ is a rt. $\angle$ (§ 210). Use § 305.

7. $\angle APQ$ is a rt. $\angle$ (§ 238). Use § 305.

PAGE 211

8. $\angle ABD = \angle APC$ (each $\stackrel{m}{=} \frac{1}{2}\overset{\frown}{AC}$, § 235). Also, $\angle BAD = \angle PAC$ (Hyp.). Use § 305.

9. Draw a pair of corresponding diagonals in the given rectangles. Then each pair of corresponding ▲ in the two rectangles are similar (§ 310). Hence, the rectangles are similar (§ 313).

10. In the $\odot CAB$, $\angle BAD \stackrel{m}{=} \frac{1}{2}\overset{\frown}{AB}$ (§ 241). Also $\angle C \stackrel{m}{=} \frac{1}{2}\overset{\frown}{AB}$ (§ 235). $\therefore \angle BAD = \angle C$. In like manner, $\angle CAB = \angle D$. $\therefore$ the ▲ are similar (§ 305).

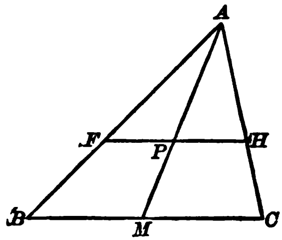
1. In the similar ▲ ACD and BFC, $AD : BF = AC : BC$ (§ 302). $\therefore AD \times BC = BF \times AC$ (§ 278), etc.

2. By Ex. 5, p. 210, ▲ APC and AFC are similar, etc. $CP \times CF = \overline{CA}^2$ (§ 278): CA is constant. $\therefore CP \times CF$ is constant.

3. See Ex. 4, p. 210.

4. $\triangle$ ABC and PBC are similar by Ex. 3, p. 210, etc.

5.



Let AM be a median of $\triangle ABC$, $FH \parallel BC$. Let FH intersect AM in P. Then by means of similar $\triangle$ show that $\frac{BM}{FP} = \frac{AM}{AP} = \frac{MC}{PH}$. $\therefore FP = PH$ (§ 282).

PAGE 212

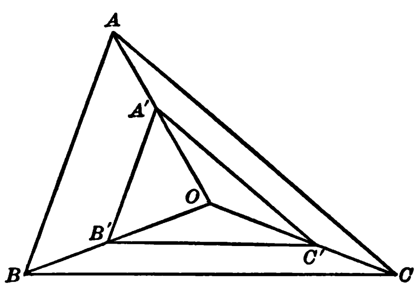
6. The $\triangle$ APF and FQB are similar (§ 305).
$\therefore PF : FQ = AF : FB$ (§ 302). But $\frac{AF}{FB}$ is constant, etc.

7. $\angle ABF = \angle FBC$ (measured by ½ of equal arcs (§ 235)). $\therefore$ in the $\triangle ABC$, $AB : BC = AE : EC$ (§ 301)

9. Use § 324.

10. $\frac{PQ}{BC} = \frac{AP}{AB} = \frac{DT}{DC} = \frac{RT}{BC}$ (§§ 294, 295). Use § 282.

11.



$\triangle$ ABO and $A'B'O$ are similar (§ 305). $\triangle$ $OB'C'$ and OBC are similar (§ 305). $\therefore A'B' : AB = (OB' : OB) = B'C' : BC$ (§ 302).

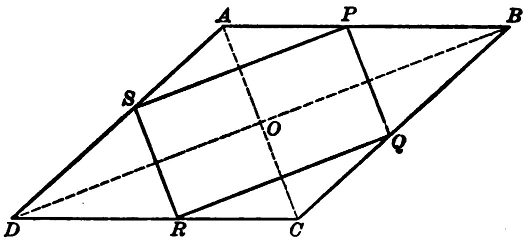
But $\angle A'B'C' = \angle ABC$ (§ 112). $\therefore$ ▲ $A'B'C'$ and ABC are similar (§ 310).

12. $\dfrac{AR}{RQ} = \left(\dfrac{BR}{RD}\right) = \dfrac{RP}{AR}$ (§ 302). $\therefore \overline{AR}^2 = RQ \times RP$ (§ 278).

1. Then use §§ 101, 146.

2. Use Ex. 8, p. 93, and Ex. 1, p. 212.

3.



Let $ABCD$ be the given rhombus, and P, Q, R, S be the midpoints of its sides. Then $PQRS$ is a ▱ (Ex. 1, p. 212). But $AC \perp DB$ (§ 121), $SP \parallel DB$, $PQ \parallel AC$. $\therefore \angle SPQ = \angle AOB$ (§ 112). In like manner $\angle PQR =$ a rt. $\angle = \angle QRS = \angle RSP$. $\therefore PQRS$ is a rectangle (§ 149).

4. Use § 305.

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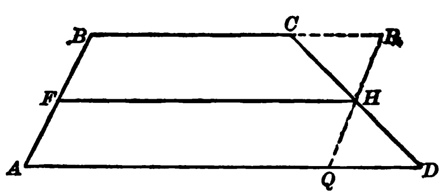
5. Draw the altitude AH. Then the ▲ AHC and BFC are similar (§ 305). $\therefore AC : HC = BC : FC$ (§ 302). But $HC = \frac{1}{2} BC$ (§ 118). $\therefore AC : \frac{1}{2} BC = BC : FC$ (Ax. 9). $\therefore AC \times FC = \frac{1}{2} \overline{BC}^2$ (§ 278). Or, $2\, AC \times FC = \overline{BC}^2$ (Ax. 4).

6. Draw the chord BD and prove the ▲ ABE and ABD similar (Ex. 5, p. 210).

7.



Let $ABCD$ be the trapezoid and F and H the midpoints of its legs AB and CD. Then $FH \parallel AD$ and BC, otherwise § 296 (last sentence) would be violated. Through H draw $PQ \parallel AB$ and meeting

AD in Q and BC produced at P. **Prove** $\triangle CHP = \triangle HQD$ (§ 80). $\therefore CP = QD$. $FH = AQ$ (§ 155) $= AD - QD$. $FH = BP$ (§ 155) $= BC + PC$. $\therefore$ $2FH = AD + BC$, etc.

8. Denote the tangent by RTS. Then $\angle P = \angle QTR = \angle STQ' = \angle P'$ (§§ 235, 241), etc.

9. Let the point P lie between Q and R. $\angle RQP = \angle RTP = \angle P'TR' = \angle P'Q'R'$ (§§ 235, 241), etc.

10. Draw the diameter AR intersecting CD in F. ▲ AFQ and ARP are similar (§ 305). $\therefore$ $AR : AP = AQ : AF$. $\therefore$ $AP \times AQ = AR \times AF$ (constant), etc.

11. In the similar ▲ ABF and DBC, $AB : AF = BD : CD$. $\therefore$ $AB \times CD = AF \times BD$. In sim. ▲ BCF and ABD, $BC : FC = BD : AD$. $\therefore$ $BC \times AD = CF \times BD$. Adding, $AB \times CD + BC \times AD = (AF + FC)\, BD = AC \times BD$.

1. (Group 54). In the similar ▲ ABF and BFC, $AB : AF = BC : BF$, etc.

2. $\overline{AB}^2 = AC \times AF$, and $\overline{BC}^2 = AC \times FC$ (§ 319), etc.

3. Draw the chord QB. Then ▲ AQB and APB are similar (§ 305), etc.

4. See Ex. 2, p. 199. **5.** Use §§ 69, 238, 304. **6.** Use §§ 96, 235, 304.

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7. $\angle AFB$ is a rt. $\angle$ (§ 238). **Prove** ▲ DAB and ABC each similar to $\triangle AFB$ (§ 305), hence to each other (§ 306), etc.

9. § 323.

10. Draw the line of centers and the radii to the points of contact. **Prove** the ▲ thus formed similar (§§ 69, 82, 305), etc.

11. The line joining the vertex to the midpoint of the base (See Ex. 5, p. 211).

12. Denote the center by O. **Prove** $\angle POR$ a rt. $\angle$. (For $\angle POA = \angle POQ$, etc.) Use § 319.

13. $OA \times OB = \overline{OP}^2$ (Hyp.). $\therefore$ ▲ OPA and OPB are similar (§ 310).

14. By sim. ▲, $HP : FP = CH : FB = CH : AF = CK : AK = HK : FK$.

1. Construct a fourth proportional to c, a, b; to $2\,c, a, b$ (§ 297).

2. § 321. Construct a mean proportional between $a + b$ and $a - b$.

3. § 321. Construct a mean proportional between $3\,a$ and b.

4. § 321; §§ 321, 128. Construct a mean proportional between 1 and 3. To construct $\frac{1}{2}\sqrt{5}$, construct a mean proportional between 1 and 5 and bisect the result.

PAGE 215

5. *I.e.*, proportional to 12, 3, 4 (§ 298).
6. § 301. That is, bisect the ∠ of the △ opposite the side to be divided.
7. §§ 321, 298. That is, find a mean proportional between 1 and 2 (§ 321). Then use § 298.
8. These ⩓ are = (§ 80). ∴ to construct, draw $PL \parallel AC$ and meeting BC at L, bisect CL, etc.
9. $PQ = QR$. ∴ $OQ \perp PR$ (§ 203). ∴ to construct PQ, on OP as a diameter construct a semicircle, etc. Use § 238.
10. Use § 323.
11. They are equal (§ 301). ∴ divide the chord AB into line segments which shall be as 2 : 3 (§ 298) and on AQ construct a segment of a ⊙ which shall contain an ∠ = $\frac{1}{2}$ ∠ inscribed in segment APB (§ 265).
12. See Ex. 9, p. 169.
13. § 316.
14. Sides of required △ = chords subtended by inscribed ∡ = ∡ of given △.
15. Suppose the construction made and lines drawn from the center to the points of contact. Then ∡ at the center = supplements of ∡ of given △. Hence to construct, at the center of the given ⊙ construct adj. ∡ = supplements of the ∡ of the given △. At the points where the radii which separate these ∡ meet the circle draw tangents to the circle.
16. Draw a line bisecting two opposite sides of the rectangle. Or let $ABCD$ be the given rectangle. On AD, its lower base, describe a semicircle intersecting BC in F. Draw $FH \perp AD$. Then FH divides the rectangle as required. For $AH : HF = HF : HD$ (§ 320). Use Ex. 9, p. 211. In case the semicircle does not reach the side BC, the latter method of construction cannot be used.

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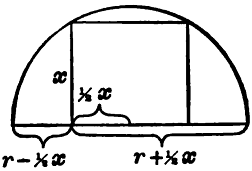
17. By similar ⩓, $AH : DE = (AB : DB) = AK : DG$. But $DE = DG$. ∴ $AH = AK$ (§ 282). Hence,

CONSTRUCTION. Draw AK the altitude of the given $\triangle$. From A draw $AH \parallel BC$ and $= AK$. Draw HB intersecting AC in E. Draw $ED \parallel BC$ and meeting AB in D. Draw DG and $EF \perp BC$, etc.

1. Use the method of the Ex. in § 327.

2. Hence, $x = -a \pm a\sqrt{2}$. Construct $a\sqrt{2}$, then $a\sqrt{2} - a$; use Ex. 12, p. 139.

3.



Denote a side of the required square by x. Then $r - \frac{1}{2}x : x = x : r + \frac{1}{2}x$ (§320). $\therefore x^2 = r^2 - \frac{1}{4}x^2$, or $x^2 = \frac{4r^2}{5}$, or $x = \frac{2}{5}\sqrt{5r^2}$. $\therefore$ to construct x, construct a mean proportional between r and $5r$ and take $\frac{2}{5}$ of the result.

4. Denote the first given line by a, the other given line by b, and the required part by x. Then $a - x : x = x : b$.

$\therefore x = \frac{-b \pm \sqrt{b^2 + 4ab}}{2}$.

Hence, construct a mean proportional between b and $b + 4a$, etc.

PAGE 217

1. Divide the given line AB in the ratio of $m : n$ (§ 298).

2. n m d

Let d be the difference of the two required lines and m and n two other lines in the given ratio. Denote the smaller of the required lines by x. Then the other line required is $d + x$. Hence $m : n = d + x : x$ (Hyp.) or $m - n : n = d : x$ (§ 287). Hence construct a 4th proportional to $m - n$, n, and d, etc.

3. Use the figure of Ex. 10, p. 212 (text-book). Let $ABCD$ be the given trapezoid and PT the line $\parallel$ the base BC which divides $ABCD$ into two similar trapezoids $APTD$ and $PBCT$. Then $AD : PT = PT : BC$ (§ 302). Hence,

CONSTRUCTION. Construct the mean proportional between AD and BC (§ 321). From any point X in AB draw a line $XY \parallel AD$ and $=$ this mean proportional. Through Y draw a line $\parallel AB$ and meeting CD in T. From T draw $TP \parallel BC$, etc.

Proof. The trapezoids $APTD$ and $PBCT$ are mutually equiangular (§ 97). Let AB and CD be produced and meet at O. Then $OA : OP = (AD : PT = PT : BC) = OP : OB$. $\therefore OA - OP : OA = OP - OB : OP$. $\therefore AP : OA = BP : OP$. $\therefore AP : BP = (OA : OP) = AD : PT$, etc.

4. Let AB be the smaller given line. Produce it through B to C, making $AC =$ larger given line. Through C and B pass any convenient circle. From A draw a tangent to $\odot$ (§ 264), viz. AD. Then $\overline{AD}^2 = AB \times AC$ (§ 324).

5. Construct by drawing a line from one point A through the other, B, to meet given line at P. Find the mean proportional between PA and PB (§ 321). On the given line lay off $PC =$ this mean proportional. Through A, B, and C pass a circle (§ 256). (Another solution is obtained by laying off PC' in the opposite direction from C.)

6. Denote the external segment by x. From the given point draw a tangent $(= a)$. Then $2x : a = a : x$ (§ 324). Use § 327.

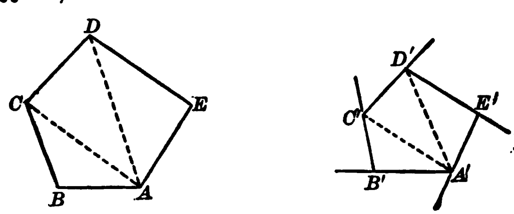
7. $m^2x^2 = \dfrac{mt^2}{m+n}$. $\therefore t : mx = mx : \dfrac{mt}{m+n}$.

Given the lines m, n, t, find $\dfrac{mt}{m+n}$ by constructing the 4th proportional to $m+n$, m, t. Find mx by § 321.

8. From the point of division so made draw a line to P. Through P draw a $\perp$ to the last line, etc. Use § 295.

9. Through the points of division on PQ and QR draw a line, etc. Use §§ 305, 302.

10.



Given the polygon $ABCDE$ and the line $A'B'$.

To construct on $A'B'$ a polygon similar to $ABCDE$ and similarly placed.

CONSTRUCTION. In the given polygon, draw the diagonals AC and AD, dividing the polygon into triangles. At B', on the line $A'B'$, construct $\angle A'B'C'$ equal to $\angle B$; and at A' construct $\angle B'A'C'$ equal to $\angle BAC$ (§ 86). Produce the lines $B'C'$ and $A'C'$ to meet at C'. $\therefore \triangle ABC \sim \triangle A'B'C'$ (§ 305). Proceed in like manner, using § 305 twice again, and § 313.

11. Divide the given line into parts in the given ratio by § 295. Upon the given line as a diameter construct a circle. Use § 265. Or, construct a diameter $\perp$ given line, and a chord from end of second diameter through the point of division of the first, etc.

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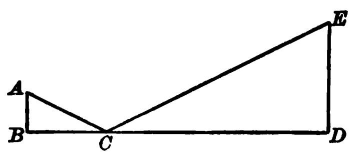
1. AB, BC, AD, so as to be able to use the proportion $AB : BC = AD : DE$.

2.



Let AB be the observer, ED the tree, and let the mirror be at C. Then $\angle ACB = \angle ECD$ (Ex. 2, p. 105). $\therefore$ the $\triangle$ are similar (§ 305). $\therefore BC : AB = CD : ED$, or $6 : 5\frac{1}{2} = 120 : ED$. $\therefore ED = 110$ ft. *Ans.*

3. $\angle FBC = \angle EAB$ (§ 114). Denote the height of the tree by x. $\therefore x : 150 = 4\frac{1}{2} : 6$. $\therefore x = 112\frac{1}{2}$ ft. *Ans.*

4. See Ex. 9, p. 197 (text-book). Construct the diagram to scale and on it measure AB, and hence, by use of a proportion, find the distance represented by AB.

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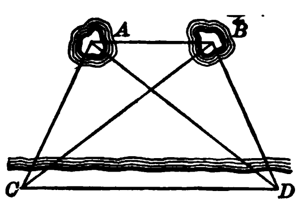
5.



Let C and D be accessible, A and B inaccessible. Measure CD and the angles ACD, BCD, ADC, BDC. Then construct a drawing to scale, and on the drawing measure AB, and then compute the distance represented by AB.

6. $\therefore \frac{\overline{FB}^2}{\overline{FA}^2} = \frac{1}{2}$. $\therefore \frac{FB}{FA} = \frac{1}{\sqrt{2}} = \frac{1}{1.41+} = \frac{5}{7}$ approx.

7. $\therefore SR = 60$. $\therefore$ 60 lb. *Ans.*

8. The diagram of Ex. 7 would need to be changed so that $BC = 18$, and $AC = 100$. $\therefore AB = \sqrt{\overline{100}^2 + \overline{18}^2} = 101.6 +$ ft. $\therefore 101.6 : 18 = 1800$ lb. $: x$. $\therefore x = 318.8 +$ lb. *Ans.*

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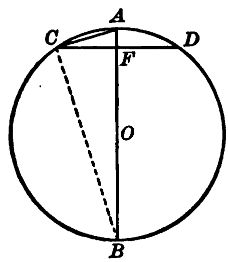
11.



Let O be the center of the earth, and CAD be a circle through the tops of the three stakes. Then AB is $\perp$ bisector of the chord CD (§ 205). $\therefore$ by § 320, $AF : AC = AC : AB$. $\therefore AF = \frac{\overline{AC}^2}{AB} = \frac{4\overline{AC}^2}{4AB} = \frac{(2AC)^2}{4 \text{ (diam. of earth)}}$. Hence, AF varies as the square of $2AC$. But for a small arc, $2AC$ approximately = arc CAD. Bulge for 4 miles = $2^2(\frac{2}{3}$ ft.) = $2\frac{2}{3}$ ft. *Ans.*
Similarly, $10\frac{2}{3}$ ft.; $42\frac{2}{3}$ ft.; $170\frac{2}{3}$ ft. *Ans.*

12. Denote the distance in miles by x. Then $x^2 : 2^2 = 11 : \frac{2}{3}$. $\therefore x = 8 +$ miles. *Ans.*

13. 16 mi.+.

14. 209.75.

15. 44; 91; 687 −; 1418 −.

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16. Let the $\perp$ from B meet AD at F, and that from D meet AC at H. Then $AF : 6000 = 1 : 1.305$. $\therefore AF = 4597 +$. $BF : 6000 = 1 : 1.556$. $\therefore BF = 3856 +$. $3856 + : FD = 1 : .839$. $\therefore FD = 3235 +$, etc. $CD = 4166 +$. *Ans.*

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5. Not unless the polygons have three sides.

6. Not unless the polygons have three sides.

7. §§ 320, 322.

8. §§ 324, 325.

9. §§ 302, 312, 313, 317, 318.

10. §§ 293, 294, 296, 299, 300, 301, 302, 304, 305, 306, 309, 310, 311, 316, 319.

11. Each line on the object is fifty times as long as the corresponding line on the drawing; 1200 times as long.

12. Two.

14. Three.

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3. 4.	5. 24.	7. 32.	9. 27.
4. 24.	6. $13\frac{1}{2}$.	8. 27.	10. 60.

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17. $13\frac{1}{2}$ sq. yd.; $13\frac{1}{2}$ sq. mi.; 1350 sq. mi.

PAGE 227

1. 16 : 3.

2. 1944.

3. 75 : 256.

4. $8\frac{1}{3}$ ft.

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1. $21\frac{7}{8}$ yd.

2. 17.5.

3. 9 sq. in.

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1. 240 sq. in.

2. 20.3 ft.

PAGE 231

1. 75.917 sq. in.

2. 2 ft.

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1. $1\frac{1}{2}$ sq. ft.

2. 16 in.; 32 in.

1. (Group 61). 35 sq. in.; 35 sq. in.

2. Area of $\triangle PQT = \frac{1}{2}PT \times h = \frac{1}{2}TR \times h =$ area $\triangle TQR$ (§ 343, Ax. 9, § 343).

3. Use §§ 159, 343, Ax. 1. 4. 1 ft. 4 in. 5. 4.8 ft.

6. In the world about us a vastly larger number of rectangles occur than of equilateral triangles; and it is easier to resolve rectangles into squares than into equilateral triangles.

7. $831\frac{1}{4}$ sq. ft.; 1360 sq. ft.; 1275 sq. ft.

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1. 962 sq. in. 3. 2.24 ft. 5. 168 sq. ft.

2. 9. 4. $\frac{2K - b'h}{h}$. 6. The base and altitude.

7. The base and altitude, or the three sides. See Formula 5, under *Areas*, p. 294.

8. The two bases and the altitude.

9. Yes. 11. No. 12. 447 sq. in. 13. $\frac{8^2 \times 4}{4^2 \times 8} = \frac{8}{4} = \frac{2}{1}$.

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14. Draw QA and $SB \perp PT$.
Prove $\triangle QAR = \triangle PBS$ (§ 110). $\therefore QA = SB$.

15. 80; 8000 sq. ft.

1. 16 : 25.

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1. 13 in. 2. 15 ft.

3. Other side of rectangle $= \sqrt{\overline{20}^2 - \overline{16}^2} = 12$. $\therefore$ 192 sq. in. *Ans.*

4. Let x = side of square. $\therefore x^2 + x^2 = 100$. $\therefore x^2 = 50$. *Ans.*

5. Alt. $= \sqrt{\overline{24}^2 - \overline{12}^2}$ in. (§ 356) $= 12\sqrt{3}$ in. $= 20.78+$ in. *Ans.*

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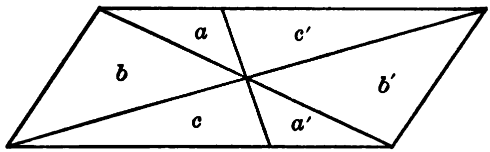
6. By the method of Ex. 5, p. 238, find altitude of $\square = 5\sqrt{3}$. $\therefore$ Area $= 100\sqrt{3}$ sq. ft. $= 173.2+$ sq. ft. *Ans.*

7. Let 24 in. be the base. Denote the altitude by x. Then $x^2 + x^2 = 18^2$. $\therefore x = 9\sqrt{2}$. $\therefore$ Area $= 108\sqrt{2}$ sq. in. or $152.73+$ sq. in. *Ans.*

8. $a\sqrt{b^2 - a^2}$.

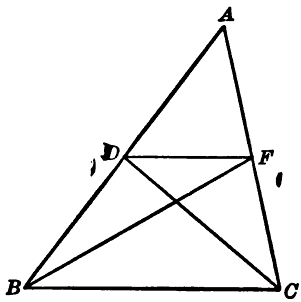
9. Distance $= \sqrt{\overline{60}^2 + \overline{18}^2}$ ft. $= 62.64$ ft. *Ans.*

1. (Group 63).



Show that $\triangle a = \triangle a'$ (§ 80), $\triangle b = \triangle b'$ (§ 79), $\triangle c = \triangle c'$ (§ 80). Add, etc.

2.



$DF \parallel BC$ (§ 300) $\therefore \triangle ABC \sim \triangle ADF$ (§ 305) $\therefore$ by § 352 $\dfrac{\triangle ABC}{\triangle ADF} = \dfrac{\overline{AC}^2}{\overline{AF}^2} = \dfrac{(2\overline{AF})^2}{\overline{AF}^2} = \dfrac{4}{1}$. Or, in $\triangle ABC$, let $AB = 2AD$, $AC = 2AF$. $\dfrac{\triangle ABC}{\triangle ADC} = \dfrac{2}{1}$ (§ 345). $\dfrac{\triangle ADC}{\triangle ADF} = \dfrac{2}{1}$ (§ 345). $\therefore \dfrac{\triangle ABC}{\triangle ADF} = \dfrac{4}{1}$ (Ax. 4).

3. See Ex. 2, and § 156.

4. See figure of Ex. 2. Show that the $\triangle$ ADF, DFB, and DFC are all equal in area.

5. Use figure, p. 206 (text-book). Area of $\triangle ABC = \frac{1}{2}AB \times BC$ (§ 343) or $= \frac{1}{2}AC \times BF$ (§ 343). $\therefore \frac{1}{2}AB \times BC = \frac{1}{2}AC \times BF$ (Ax. 1). $\therefore AB \times BC = AC \times BF$ (Ax. 4).

6. $\frac{1}{2}b \times a = \frac{1}{2}b' \times 3a$ (Hyp.). $\therefore b' : b = a : 3a = 1 : 3$ (§ 281).

7. Denote one of the legs by c and half the base of the first $\triangle$ (and $\therefore$ the altitude of the second $\triangle$) by b. Then the area of each $\triangle = b\sqrt{c^2 - b^2}$ (§§ 356, 343).

9. Two trapezoids are formed whose corresponding bases and whose altitudes are equal. Thus,
$\frac{1}{2}(\frac{1}{2}b + \frac{1}{2}b')h = \frac{1}{2}(\frac{1}{2}b + \frac{1}{2}b')h$ (§ 350).

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10. Area of $\triangle AOB$ = area of $\triangle AOD$ (§ 344). Area of $\triangle BOC$ = area of $\triangle DOC$ (§ 344). $\therefore$ area of $\triangle ABC$ = area of $\triangle DAC$ (Ax. 2).
If one diagonal of a parallelogram bisects the other diagonal, the first diagonal divides the parallelogram into two equivalent triangles.

11. $\triangle ABC = \triangle ADC$ (1), $\triangle PTC = \triangle PRC$ (2), $\triangle AQP = \triangle ASP$ (3) (§ 156). Add (3) and (2) and subtract the result from (1) (Axs. 2, 3).

12. Let $ABCD$ be the given quadrilateral, and P the midpoint of the diagonal AC. Then $\triangle ABP = \triangle PBC$ (§ 344). Also $\triangle ADP = \triangle PDC$ (§ 344). Adding, quad. $ABPD$ = quad. $PBCD$ (Ax. 2).

1. (Group 64). See figure of Ex. 1 (Group 51), p. 210 (text-book). $\overline{AB}^2 = \overline{AD}^2 + \overline{DB}^2$ (§ 355). Also $\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2$ (§ 355). Subtract.

2. Let $ABCD$ be the given quadrilateral and let the diagonals AC and BD intersect at right $\angle$s at the point O. Then, by § 355, $\overline{AB}^2 = \overline{BO}^2 + \overline{OA}^2$, $\overline{CD}^2 = \overline{OC}^2 + \overline{OD}^2$. $\therefore \overline{AB}^2 + \overline{CD}^2 = \overline{BO}^2 + \overline{OA}^2 + \overline{OC}^2 + \overline{OD}^2$. Also $\overline{BC}^2 = \overline{BO}^2 + \overline{OC}^2$, $\overline{AD}^2 = \overline{AO}^2 + \overline{OD}^2$. $\therefore \overline{BC}^2 + \overline{AD}^2 = \overline{BO}^2 + \overline{OA}^2 + \overline{OC}^2 + \overline{OD}^2$. Use Ax. 1.

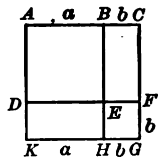
3. The altitude bisects the base (§ 117). Denote one side of the $\triangle$ by a, and the altitude by x. $\therefore a^2 = x^2 + \left(\frac{a}{2}\right)^2$ (§ 355).
$\therefore x^2 = \frac{3a^2}{4}$.

4. AB is the hypotenuse. $\therefore \overline{AB}^2 = \overline{AC}^2 + (2\overline{CK})^2 = \overline{AC}^2 + 4\overline{CK}^2$. Also $\overline{AK}^2 = \overline{AC}^2 + \overline{CK}^2$ (§ 355). Subtract, etc.

5. $\overline{AQ}^2 = \overline{AC}^2 + \overline{CQ}^2$, $\overline{BP}^2 = \overline{PC}^2 + \overline{BC}^2$ (§ 355). Adding, $\overline{AQ}^2 + \overline{BP}^2 = \overline{AC}^2 + \overline{BC}^2 + \overline{PC}^2 + \overline{CQ}^2$ (Ax. 2) $= \overline{AB}^2 + \overline{PQ}^2$ (Ax. 9).

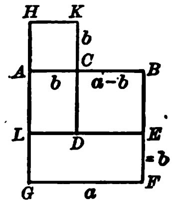
6. $\overline{BE}^2 = \overline{AB}^2 + \overline{AE}^2 = \overline{AB}^2 + \frac{1}{4}\overline{AC}^2$ (§ 355). $\overline{CF}^2 = \overline{AF}^2 + \overline{AC}^2 = \frac{1}{4}\overline{AB}^2 + \overline{AC}^2$ (§ 355). $\therefore \overline{BE}^2 + \overline{CF}^2 = \frac{5\overline{AB}^2}{4} + \frac{5\overline{AC}^2}{4}$ (Ax. 3). $4(\overline{BE}^2 + \overline{CF}^2) = 5(\overline{AB}^2 + \overline{AC}^2) = 5\overline{BC}^2$ (Ax. 9).

7.



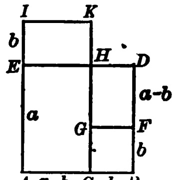
Let line AB be denoted by a, BC by b; let AG be the square on AC, and AE the square on AB. Then $EG = b^2$, and rectangles BF and EK each $= ab$. $\therefore$ $BF = EK$. But $AG = AE + EG + 2BF$, or $(a + b)^2 = a^2 + b^2 + 2ab$.

8.



Let $AB = a$, $AC = b$. Then $BC = a - b$. Let AF be the square on AB, CE the square on CB, and AK the square on AC. Hence, the rectangles LK and GE each $= ab$. Then $BD = AF + AK - 2GE$, or $(a - b)^2 = a^2 + b^2 - 2ab$.

9.



Let $AB = a$, $BC = b = IE$; then $DF = a - b$. Let AD be the square on AB, and CF the square on CB. Then the rectangle $EK = DG$ (for each $= b(a - b)$). But rectangle $AK = AD + EK - DG - CF = AD - CF$. $\therefore$ $(a + b)(a - b) = a^2 - b^2$.

10. 1 : 6.

11. 31,875 sq. mi.

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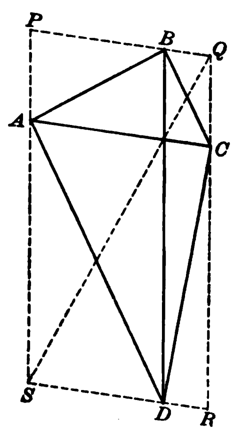
1. Draw the altitude of the ▱ through P and $\perp AD$, and denote it by h. Then by § 343 and Ax. 2, $\triangle PAD + \triangle PBC = \frac{1}{2}AD \times h = \frac{1}{2}$ ▱ $ABCD$ (§ 339), etc.

2. Draw a $\perp$ from P to AD. Use § 355 (four times), Ax. 2, etc.
3. See method of proof in § 386.
4. Show that the top and bottom $\triangle$ together $= \frac{1}{2}$ the trapezoid, *i.e.* $= \frac{1}{4}h(b_1 + b_2)$.
5. See Ex. 4.
6. $\overset{\frown}{AC} + \overset{\frown}{BD} =$ semicircle (for $\frac{1}{2}$ their sum measures the rt. $\angle\ O$). Also $\overset{\frown}{BD} + \overset{\frown}{DE} =$ semicircle (constr.). $\therefore \overset{\frown}{AC} + \overset{\frown}{BD} = \overset{\frown}{BD} + \overset{\frown}{DE}$ (Ax. 1). $\therefore \overset{\frown}{AC} = \overset{\frown}{DE}$ (Ax. 3). $\therefore$ chord $AC =$ chord DE (§ 200). $\therefore \overline{OA}^2 + \overline{OC}^2 =$ (chord $AC)^2 =$ (chord $ED)^2$ (§ 355). But $\overline{OB}^2 + \overline{OD}^2 =$ (chord $BD)^2$ (§ 355). $\therefore \overline{OA}^2 + \overline{OC}^2 + \overline{OB}^2 + \overline{OD}^2 =$ (chord $ED)^2 +$ (chord $DB)^2 = \overline{EB}^2$.
7. Denote the vertices of the quadrilateral by A, B, C, D, and the midpoints of AB, BC, CD, DA by 1, 2, 3, 4, respectively. Let the diagonals BD and AC intersect in O. Draw $O1$. Let 12 intersect BD in P. Then $OP = PB$ (§ 296). $\therefore \triangle\ O1P = \triangle\ 1PB$ (§ 344). Let 14 intersect AO in Q. Then similarly $\triangle\ O1Q = \triangle\ A1Q$, etc.
8.



Let $ABCD$ be the given quadr. Through B draw $PQ \parallel$ the diagonal AC, through D draw $SR \parallel AC$, through C draw $QR \parallel BD$, through A draw $PS \parallel BD$. Then show $ABCD = \frac{1}{2}PQRS$. Also $\triangle\ SPQ = \frac{1}{2}PQRS$ (§ 156). $\therefore ABCD = \triangle\ PQS$. But $PQ = AC$ (§ 157), etc.

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1. The base of $\square$ = median of the trapezoid (§ 157). Use § 339, Ex. 7, p. 213.
2. Join the vertex with the point in base. Use two pairs of $\triangle$s equal in area. Add.
3. Draw $\perp$s from the other vertices of the $\square$, viz.: B and D, to the diagonal AC. See Ex. 11, p. 93.
4. The altitude $= \sqrt{a^2 - \frac{a^2}{4}} = \frac{a\sqrt{3}}{2}$ (§ 356), etc.
5. By Ex. 4, if the side of the second $\triangle$ is a, the side of the first $\triangle$ is $\frac{a\sqrt{3}}{2}$. $\therefore K : K' = \left(\frac{a\sqrt{3}}{2}\right)^2 : a^2$ (§ 352) $= \frac{3a^2}{4} : a^2 = 3 : 4$.
6. Denote side of $\triangle$ by a and draw lines from the given point to the vertices of the $\triangle$. Then area of $\triangle = \frac{1}{2}a \times$ (sum of $\perp$s) (§ 343, Ax. 2). But area of $\triangle = \frac{1}{2}a \times h$ (§ 343). $\therefore \frac{1}{2}a \times$ (sum of $\perp$s) $= \frac{1}{2}a \times h$ (Ax. 1). $\therefore$ sum of $\perp$s $= h$ (Ax. 5).
7. See Ex. 10, p. 240.
8. In the $\triangle$ in which the included angle is obtuse, produce the base through the vertex of the obtuse angle, till the produced part = base, etc.
9. Use § 345, Ax. 4.
10. $\triangle QDC = \frac{1}{2} \square ABCD$ (same base CD, etc.). $\triangle ABP + \triangle PCD = \frac{1}{2} \square ABCD$, etc. Use Ax. 3.

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1. Construct a right triangle whose legs are the sides of the given squares.
2. Reduces to Ex. 12, p. 139.
3. See Ex. 1.
4. See Ex. 1.
5. See Ex. 1.
6. Draw a line through the vertex $\parallel$ the base. Bisect base; at the midpoint of the base erect a $\perp$ to the base, etc.
7. Draw a line through the vertex $\parallel$ to the base. At one end of the base construct an $\angle$ = given $\angle$, etc.
8. Draw a line through the vertex $\parallel$ the base. Using one end of the base as a center and the required side as a radius, describe an arc, etc.
9. See Ex. 7.

10. Let b be the base of the given $\triangle$, and x that of the required $\triangle$. $\therefore\ b^2 : x^2 = 1 : 2.\quad \therefore\ x = b\sqrt{2}$ (see Ex. 2, p. 208).

11. Let ABC be the given triangle. Construct any square and draw its diagonal. Then divide the side BC into segments at G such that $BC : BG$ = diagonal of square : side of square. Through G draw a line $\parallel AC$, etc.

12. Draw the $\perp$ through the intersection of the diagonals. (See Ex. 1, p. 239.)

13. Draw the line through the given point and through the intersection of the diagonals. (See Ex. 1, p. 239.)

14. Use § 321.

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1. $\dfrac{2 \times 36}{4 \times 6} = \dfrac{3}{1}$. *Ans.* $\dfrac{b''}{a''}$. *Ans.* 2. 4.

3. Since 6 = radius of the log, a side of the square beam $= \sqrt{6^2 + 6^2} = 6\sqrt{2}$. On the diagram of Ex. 6, p. 219, $AB = 12$, $DB = 4$. $\therefore\ FB = \sqrt{4 \times 12} = 4\sqrt{3}$ (§ 320). $FB : AF = 1 : \sqrt{2}$, or $4\sqrt{3} : AF = 1 : \sqrt{2}$. $\therefore\ AF = 4\sqrt{6}$. $\dfrac{(4\sqrt{6})^2 \times 4\sqrt{3}}{(6\sqrt{2})^2 \times 6\sqrt{2}} = \dfrac{8\sqrt{3}}{9\sqrt{2}} = \dfrac{13.85+}{12.72+}$. *Ans.*

4. When the width of the beam is 3 in., by § 320 we find height of beam $= 3\sqrt{15}$. $\therefore$ Required ratio $= \dfrac{(4\sqrt{6})^2 \times 4\sqrt{3}}{(3\sqrt{15})^2 \times 3} = \dfrac{128\sqrt{3}}{135} = \dfrac{221.69+}{135}$. *Ans.*

5. Denote the offsets by $a_1, a_2, a_3, \ldots a_n$, and the common distance between them by h. Then the sum of the areas of the small trapezoids forming the figure $= \frac{1}{2}h(a_1 + a_2) + \frac{1}{2}h(a_2 + a_3) + \ldots + \frac{1}{2}h(a_{n-1} + a_n) = \frac{1}{2}h(a_1 + a_n) + h(a_2 + a_3 + \ldots + a_{n-1})$.

6. In Ex. 5, if the initial and final offsets reduce to zero the formula becomes $h(a_2 + a_3 + \ldots + a_{n-1}1)$. In the diagram of Ex. 6 draw the longest possible chord to the irregular curve and divide this chord into a suitable number of equal parts. At the points of division erect $\perp$s to meet the curve. Then the entire area approximately equals the sum of the $\perp$s multiplied by the common distance between them.

7. Let $FG = 100$ ft., $GK = 140$ ft., $KL = 80$ ft., $FB = 160$ ft., $GC = 220$ ft., $KD = 220$ ft., $LE = 140$ ft., $HA = HG = 50$ ft. Then area of $ABCD = \frac{100}{2}(160 + 220) + 140 \times 220 + \frac{80}{2}(220 + 140) - \frac{150}{2}(160 + 50) - \frac{170}{2}(50 + 140) = 32{,}300$ sq. ft. *Ans.*

8. Let O be the center of $\widehat{BC}$. At B erect $BO \perp AT$. Draw Ob_3. From b_3 draw $b_3H \perp OB$. Then $OH = \sqrt{r^2 - (3d)^2}$. $\therefore a_3b_3 = BH = r - \sqrt{r^2 - (3d)^2}$. In like manner, $a_nb_n = r - \sqrt{r^2 - (nd)^2}$.

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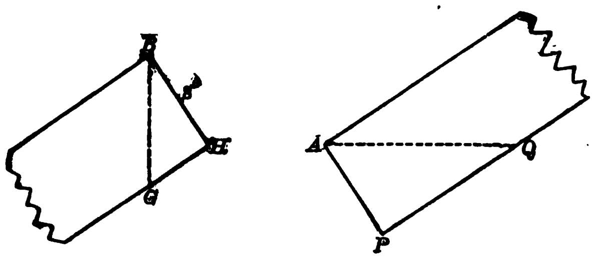
9. Speed per hour $= \sqrt{12^2 + 3^2}$ mi. $= 12.36 +$ mi.

12. 500 lb. 14. 5.66 + mi. per hour. 15. 25 mi. per second.

16. Use the tape to make a right triangle whose sides are 24 ft., 32 ft., 40 ft.

17.



$AD = 10$ ft., $BD = 6\frac{2}{3}$ ft., $\therefore$ AB $= 12.01 +$ ft.
At the peak, GH, the part cut off from the lower edge of the rafter, is $\frac{2}{3}$ of BH. To determine the line along which, at the lower end of the rafter, the lower edge of the rafter must be cut, $PQ = \frac{2}{3}\ AP$, where AP is the width of the rafter.

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3. §§ 343, 344, 345, 346, 352, 355, 356.

4. §§ 334, 335, 336, 337. 7. § 350.

5. §§ 339, 340, 341, 342, 347. 8. §§ 353, 354.

6. § 348. 9. See § 333.

10. "The number of units in the area of a rectangle divided by the number of linear units in the base."

11. When two sides of a right triangle are known, Prop. X enables us to find the third side without the labor of measuring it.

14. See p. 280.

15. Correct for square and rectangle. It gives too large a result for all other quadrilaterals.

PAGE 248

1. 9 : 64.

PAGE 251

1. 108°. 2. Use § 79.

PAGE 252

7. Side of square = $\sqrt{3^2+3^2}$ in. = $3\sqrt{2}$ in. = 4.23 + in. Area = 18 sq. in. *Ans.*

8. $r^2 + r^2 = (\frac{3}{2})^2$. $\therefore$ r = 1.06 + in. *Ans.*

9. $7\sqrt{2}$ in. = 9.89 + in. *Ans.*

10. Use § 121. 11. Use § 345. 12. § 363.

PAGE 253

1. 72°. 2. 60°. 3. 90°.

4. $5\sqrt{2}$ in. = 7.07 + in. *Ans.* Area = 200 sq. in. *Ans.*

5. Apothem = $5\sqrt{3}$ in. = 8.66 + in. *Ans.* Area $\frac{3\sqrt{3}}{2} \times 100$ sq. in. = 259.8 + sq. in. *Ans.*

6. $\frac{1}{2}b\sqrt{3}$; $\frac{3\sqrt{3}b^2}{2}$.

7. $\frac{1}{2}b\sqrt{2}$; $2b^2$.

8. 4 in.

9. $\widehat{FA} = \widehat{FB}$ (§ 202) $\therefore$ $FA = FB$ (§ 363), etc.

10. If apothem = x, $r = 2x$ (Ex. 9). $\therefore$ $(2x)^2 - x^2 = 9$. $\therefore$ $x = \sqrt{3}$ in. Area = $9\sqrt{3}$ sq. in. *Ans.*

11. See Ex. 10. $r = 4$. *Ans.*

12. Circumscribe a circle about the given polygon. Use §§ 198, 235.

13. Area = 584.56 + sq. ft.

PAGE 255

5. 400 sq. in.

6. 129.90 sq. in.; 519.61 sq. in.

7. Denote the radius of the circle by r. Then area of hexagon $= \frac{3\sqrt{3}r^2}{2} = 2\left(\frac{3\sqrt{3}r^2}{4}\right) = 2$ area of triangle.

8. $\frac{2r^2}{4r^2} = \frac{1}{2}$. *Ans.*

9. 3: 4.

10. On the diagram of p. 249, draw straight lines connecting the alternate vertices.

11. In each of at least two adjacent squares on the squared paper, inscribe a regular octagon (see Ex. 11, p. 270). The figure is completed by drawing lines parallel to the sides of these octagons, which lines will form sides of inscribed octagons in the other squares composing the squared paper.

PAGE 256

1. $128\sqrt{3}$ sq. in. = 221.70 + sq. in.
2. Use §§ 241, 102, etc.
3. Use Ex. 2 and § 352.
4. See Ex. 9, p. 253.

PAGE 257

1. 1 : 2; 1 : 4.
2. 4 : 9; 2 : 3; 2 : 3.
3. Use §§ 376, 353. ∴ 414.72 sq. in. *Ans.*
4. 16 : 25; 16 : 25.

PAGE 258

1. (§ 379). $10\sqrt{3}$.

3. 8.04 − in. *Ans.* The perimeters and their difference would each be doubled. 16.07 + in. *Ans.*

PAGE 259

1. 4 : 1.
2. $27\sqrt{3}$ sq. in. or 46.764 + sq. in.; 25.236 + sq. in.

PAGE 261

1. 88 in.
2. 66 in.
3. 11 ft.
4. $7\frac{6}{7}$ yd.
5. $10\frac{3}{7}$ ft.
6. .0993 + in.
7. Two places.
8. $r = 14$ in.
9. $r = 2.1$ in.
10. $r = \frac{1}{2}$ in.
11. $r = 9$.
12. 2 : 3 (§ 381).
13. 5 : 7.
14. $7\frac{1}{7}$ in.
15. $9\frac{1}{7}$ in.

16. $\frac{117}{360} \times 44$ in. $= 14.3$ in. *Ans.*

17. $2\frac{281}{360}$ in.

18. 8.

19. Circf. $= 132$ in. $\frac{33}{132}$ of $360° = 90°$. *Ans.*

20. $22\frac{1}{2}°$.

21. $152\frac{8}{11}°$.

22. 3.

PAGE 262

1. Make the arc greater than a semicircle.

PAGE 264

1. 616 sq. in.

2. 346.5 sq. in.

3. .000616 sq. ft.

4. .0616 sq. in.

5. $5\frac{211}{224}$ sq. in.

6. $7\frac{1}{14}$ sq. in.

7. $\frac{22a^2}{7}$ sq. ft.

8. $\frac{11a^2}{224}$ sq. ft.

9. 2464 sq. ft.

10. .1386 sq. in.

11. $\frac{504}{1331}$ sq. in.

12. $1018\frac{2}{7}$.

13. 40.26 sq. in.

14. 100 : 1; 10 : 1; 10 : 1.

15. 4 times (§ 393).

16. $56\frac{4}{7}$ in.

17. 21.

18. $4b$.

19. 20.

20. 5.64 + sq. ft.

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21. $102\frac{2}{3}$ sq. in.; $136\frac{8}{9}$ sq. in.; $35\frac{14}{15}$ sq. in.

22. πb^2; $\frac{\pi b^2}{4}$; $4\pi b^2$.

23. $\frac{\pi R^2}{4}$; $4\pi R^2$; $3\pi R^2$.

24. $\frac{\pi R^2}{16}$; $\frac{\pi R^2}{2}$; $\frac{\pi R^2}{8}$.

25. 9.00 + in.

26. Measure the circumference of the pipe and divide the result by π.

27. Circf. of the first wheel = 6 times the circf. of the second. $\therefore 6 \times 120 = 720$. *Ans.* $300 \div 6 = 50$. *Ans.*

28. It becomes $\frac{9}{4}$ of what it was (since $(1\frac{1}{2})^2 : 1^2 = \frac{9}{4} : 1$).

29. $3^2 : x^2 = 1 : 2$. $\therefore x = 3\sqrt{2} = 4.24 +$. $\therefore 4\frac{1}{4}$ in. *Ans.*

30. $\overline{12}^2 : \overline{18}^2 = 2^2 : 3^2 = 4 : 9$. $\therefore 40{,}000 \times \frac{9}{4} = 90{,}000$. *Ans.*

31. (1) $\frac{9}{16}$ of 50,000 lb. = 28,125 lb. *Ans.*
(2) $1 : x^2 = 50{,}000 : 150{,}000$. $\therefore x^2 = 3$. $\therefore 1.732 +$ in. *Ans.*

PAGE 266

1. A line $\perp$ to both the given lines (§§ 137, 100).

2. Diagonal $= \sqrt{12^2 + 5^2}$ ft. $= 13$ ft. long. *Ans.*

PAGE 267

3. Side of square $= \frac{1}{8}$ mi. $\therefore$ Area $= \frac{1}{64}$ sq. mi. $= 10$ acres. Radius of circle $= \frac{7}{88}$ mi. $\therefore$ area of circle $= \frac{7}{352}$ sq. mi. $= 12\frac{8}{11}$ acres. $\therefore$ $2\frac{8}{11}$ acres. *Ans.*

4. 36 sq. in. In the second $\triangle$, alt. $= 3\sqrt{3}$. $\therefore K = 18\sqrt{3}$ sq. in. $= 31.176 +$ sq. in.

5. 200 ft., 2400 sq. ft.; 200 ft., 2500 sq. ft.

6. $6.928 +$ sq. in.; 9 sq. in.; $10.392+$ sq. in.; $11.45 +$ sq. in. For, by Ex. 4, p. 242, $a = 4$. $\therefore K$ of $\triangle = \dfrac{16\sqrt{3}}{4} = 6.928+$. By § 362, K of a regular hexagon, whose side is a, $= \dfrac{6a^2\sqrt{3}}{4} = \dfrac{3a^2\sqrt{3}}{2}$. $\therefore K$ of given hexagon $= \dfrac{3 \times 2^2\sqrt{3}}{2} = 6\sqrt{3} = 10.392+$. For method of finding area of $\odot$ see Ex. 9, p. 264.

7. $22.33+$ in.; $19.596+$ in.; $17.05+$ in. By Ex. 4, p. 242, in given $\triangle$, $\dfrac{a^2\sqrt{3}}{4} = 24$. $\therefore a = 7.43+$. Side of square $= \sqrt{24} = 4.899+$. For circle, $\pi R^2 = 24$. $\therefore R = \sqrt{\dfrac{84}{11}}$, etc.

1. (Group 76). One; three.

2. Four; five. 3. Six.

4. Seven; eight; n; an infinite number.

5. *A B D E H I M O T U V W X Y.*

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1. No; yes. 2. No; yes.

3. Regular polygons of an even number of sides have a center of symmetry; regular polygons of an odd number of sides do not.

4. Yes.

6. Two (the diagonals); yes.

7. One diagonal; no, except when the figure is a rhombus.

8. Draw any line through this point and terminated by the perimeter. Show that it is bisected (§ 80).
9. Diameter $\perp$ the chord of the segment.
10. No; no.
11. Two; yes.
12. Line of centers.
13. *I N O X Z*; *o s x z*.
1. (Group 78). §§ 367, 235, 304.
2. Circumscribe a circle about the given pentagon (§ 367). $\angle KBC \overset{m}{=} \frac{1}{2}\widehat{EDC}$, and $\angle BKC \overset{m}{=} \frac{1}{2}(\widehat{AE} + \widehat{BC})$. $\therefore \angle KBC = \angle BKC$, etc.
3. Find side of $\triangle$ is $r\sqrt{3}$ (see Ex. 9, p. 253), etc.
4. $2r^2 - \dfrac{3\sqrt{3}r^2}{4}$ or $r^2(2 - \frac{3}{4}\sqrt{3})$.

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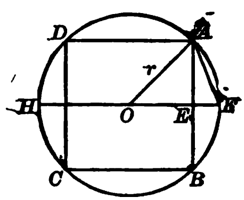
5.



In the circle O, let $ABCD$ be an inscribed square, and F the mid-point of the arc AB. $\therefore$ chord AF = side of inscribed regular octagon (§ 361). $OE = \sqrt{r^2 - \overline{AE}^2} = \sqrt{r^2 - \dfrac{r^2}{2}} = \dfrac{r}{2}\sqrt{2}$.

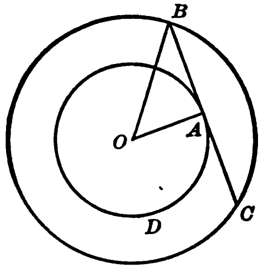
$\therefore\ EF = r - \dfrac{r}{2}\sqrt{2}$.

Then, by § 320, $AF = \sqrt{HF \times EF} = \sqrt{2r\left(r - \dfrac{r}{2}\sqrt{2}\right)}$

$$= r\sqrt{2\left(1 - \frac{\sqrt{2}}{2}\right)} = r\sqrt{2 - \sqrt{2}}.\quad \textit{Ans.}$$

6. Follow the method used in Ex. 5.
7. See Ex. 11, p. 252. Side $= \sqrt{\dfrac{a^2}{4} + \dfrac{a^2}{4}} = \dfrac{1}{2}a\sqrt{2}$. *Ans.*
8. 1 : 3.
9. The squares formed include between them six equilateral $\triangle$ each of whose sides is b. $\therefore$ area of dodecagon $= (3\sqrt{3} + 6)b^2$.
10. § 393.

11.



Let OAD and OBC be the two circles forming the circular ring. Denote the radius of the larger circle by R and that of the smaller circle by r. $\therefore$ area of ring $= \pi(R^2 - r^2)$. But in the right $\triangle$ OAB, $\overline{OB}^2 - \overline{OA}^2 = \overline{AB}^2$, *i.e.*, $R^2 - r^2 = \overline{AB}^2$. $\therefore$ area of ring $= \pi\overline{AB}^2$, etc.

12. Denote AB by q, BC by p, AC by r. The sum of the crescents $= \triangle ABC + AcB + BdC - AaBbC = \frac{1}{2}pq + \frac{\pi q^2}{8} + \frac{\pi p^2}{8} - \frac{\pi r^2}{8}$ (§ 390) $= \frac{1}{2}pq + \frac{\pi}{8}(q^2 + p^2 - r^2)$. But $r^2 = p^2 + q^2$ (§ 355). $\therefore$ sum of crescents $= \frac{1}{2}pq = \triangle ABC$.

13. $\therefore$ in like manner side BC = side DE = side AB = side DC = side AE. (Use §§ 235, 357, Ax. 3, § 200.)

14. Prove the right $\triangle$ thus formed equal by § 216, Ax. 5, § 111.

15. Denote the center by O. Then $PT = PQ$ (Hyp.), $PA = PB$ (§ 216). $\therefore$ $AT = BQ$ (Ax. 3). $\therefore$ rt. $\triangle OAT$ = rt. $\triangle OBQ$ (§ 79). $\therefore$ $\angle T = \angle Q$ (§ 216, Ax. 4). Similarly, $\angle Q = \angle S = \angle P = \angle R = \angle T$.

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1. Denote the radii of the given circles by R and r. Then the new radius $= R + r$.

2. The new radius $= R - r$.

3. The new radius $= \sqrt{R^2 + r^2}$. See Ex. 1, p. 243.

4. New radius $= \sqrt{R^2 - r^2}$. See Ex. 2, p. 243.

5. The trees must be arranged at the vertices of a regular octagon of which each side is 20 ft. By Ex. 5, p. 269, $20 = r\sqrt{2 - \sqrt{2}}$,

whence $r = 10\sqrt{2 + 2\sqrt{2}}$. Hence, if a circle is described with a radius of one inch, and a regular octagon is inscribed in the circle, the drawing will answer the given conditions and have a scale of 1 in. to $10\sqrt{2 + 2\sqrt{2}}$ ft.

6. Let x be the radius of the new circle. Then $\pi x^2 : \pi r^2 = 1 : 2$.
$\therefore\ x = r\dfrac{\sqrt{2}}{2}$, etc.

7. Produce the radii of the given sector, and draw a tangent at the midpoint of the arc of the sector. Inscribe a ⊙ in the △ thus formed (§ 259).

8. Construct any square, and join the midpoint of one side of the square with a vertex which is not one of the extremities of the side taken. At the midpoint of the chord of the given segment construct an $\angle = \angle$ formed between the line drawn in the square and the side of the square whose midpoint is taken, etc.

9. Draw the bisectors of the three ∡ of the given △. In each △ thus formed inscribe a ⊙ (§ 259).

10. Divide the given ⊙ into three equal sectors (§ 362), and inscribe a circle in each sector (Ex. 7).

11. Draw lines from the center of the given square ⊥ sides, and also to the vertices. Bisect the ∡ thus formed, etc.

12. Denote the radius of the given semicircle by R, and that of the required circle by x. Then $\pi x^2 = \frac{1}{2}\pi R^2$. $\therefore\ x^2 : R^2 = 1 : 2$.
$\therefore\ x = \dfrac{\sqrt{2}R}{2}$.

1. (Group 80). $2\pi x + 2y = 2640$, $2x + y = 1000$. Hence, $x =$ 280 ft., $2x = 560$ ft. *Ans.*

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2. Use § 385. $\dfrac{5\frac{1}{8}^\circ}{180^\circ}\left(\dfrac{22r}{7}\right) = 10$. $\therefore\ r = 110.8 +$ ft. *Ans.*

3. Limit of speed $= 5000 \times 12$ in. per minute. Circf. of wheel $= \frac{22}{7} \times 27$ in. $\therefore$ no. rev. per min. $= 60{,}000$ in. $\div$ ($\frac{22}{7} \times 27$ in.) $= 707 +$. *Ans.*

5. Point of intersection of the diagonals of the square; point of intersection of the diagonals; center of the polygon; center.

6. Center of the square.

7. Ratio $= \dfrac{\pi(3^2)3}{\pi(6^2)6} = \dfrac{27}{216} = \dfrac{1}{8}$. *Ans.*

8. The area of the circular ring which forms the cross-section of the hollow tube $= \pi(R^2 - r^2)$. Hence, the radius of a solid cylindrical beam having an equivalent cross-section would be $\sqrt{R^2 - r^2}$. Use law of strains stated in Ex. 7. Hence, ratio $=$

$$\frac{\pi(R^2 - r^2)\left(\frac{R^2 + r^2}{R}\right)}{\pi(R^2 - r^2)\sqrt{R^2 - r^2}} = \frac{\pi 7\left(\frac{16+9}{4}\right)}{\pi 7(\sqrt{7})} = \frac{25}{4\sqrt{7}} = 2.36 +. \quad \textit{Ans.}$$

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11. If $OA = r$, $OC = \dfrac{5r}{4}$. $\therefore$ area of square $= \dfrac{25r^2}{8}$, while the area of the circle is $\dfrac{22r^2}{7}$. Hence, the approximation described in Ex. 11 consists essentially in using $\dfrac{25}{8}$ as the value of π. To obtain the per cent of error, we have $\left(\dfrac{22}{7} - \dfrac{25}{8}\right) \div \dfrac{22}{7} = \dfrac{1}{56} \times \dfrac{7}{22}$ $= .0056 +$. Hence, a little more than $\frac{1}{2}$ per cent. *Ans.*

PAGE 273

1. See § 364.
2. Radius; diameter; circumference.
3. It reduces the measurement of the area of a circle to the measurement of a single straight line.
4. After we have computed the ratio of the circumference to the diameter in any one circle, by the aid of Prop. IX we know the value of this ratio in every other circle without the labor of any further computation.
   Prop. X reduces the measurement of the area of a regular polygon to the measurement of two straight lines, viz.: one side and the apothem.

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5. Circles aid (1) in the construction of regular polygons. See §§ 360, 361, 362, 364, 373, etc. (2) In determining whether a given polygon is regular. See § 359. (3) In proving the properties of regular polygons. See Ex. 2 (Group 78), p. 268.

6. (1) In proving properties of a single circle and, hence, of parts of a circle. §§ 379, 390, 394. (2) In proving properties of two or more circles. §§ 380, 382, 393.

7. § 376.

8. $d : d' = c : c'$; $r : r' = c : c'$; $K : K' = c^2 : c'^2$. (Use § 291.)

9. It is equal to it; $\frac{1}{2}$.

11. The polygons are similar (§ 376). $\therefore P : P' = a : a'$ (§ 317); also $r : r' = a : a'$ (§ 318). Ratio of apothems $= a : a'$ (§ 318). $K : K = a^2 : a'^2$ (§ 353).

12. §§ 357, 359, 360, 361, 362, 363, 364, 365, 367, 368, 369, 370, 372.

13. §§ 376, 377, 378.

14. §§ 382, 383, 384, 390, 391, 392, 397.

15. §§ 379, 380, 381, 393.

PAGE 284

1. $x = \sqrt{\overline{50}^2 - \overline{40}^2}$ ft. $= 30$ ft. *Ans.*

2. $\sqrt{\overline{12}^2 + 5^2} = 13$. *Ans.*

3. The altitude of an isosceles $\triangle$ bisects the base (§ 110). Hence, altitude $= \sqrt{5^2 - 4^2} = 3$. *Ans.*

4. See § 121. $\sqrt{\overline{17}^2 - \overline{15}^2} = 8$. Hence, the other diagonal $= 16$. *Ans.*

5. Longest chord = the diameter, or 10. The diameter through the given point is $\perp$ shortest chord (§§ 208, 136). $\therefore$ half of the shortest chord (§ 202) $= \sqrt{5^2 - 3^2} = 4$. $\therefore$ shortest chord $= 8$. *Ans.*

6. Use figure p. 118 (text-book). Let $OA = 25$ in., $AB = 48$ in. Then $AR = 24$ in. (§ 202). $\therefore OR = \sqrt{\overline{25}^2 - \overline{24}^2}$ in. $= 7$ in. *Ans.*

7. Use figure p. 122 (text-book). Draw OD and OB. Let $CD =$ 12 in., $OG = 5$ in., $AB = 10$ in. Then $GD = 6$ (§ 202). $\therefore OD =$ $\sqrt{6^2 + 5^2}$ in. $= \sqrt{61}$ in. Also $FB = 5$ in. (§ 202). $\therefore OF =$ $\sqrt{61 - 25}$ in. $= 6$ in. *Ans.*

8. $\sqrt{\overline{40}^2 - \overline{20}^2} + \sqrt{\overline{40}^2 - \overline{30}^2} = 61.09 +$. $\therefore 61.09 +$ ft. *Ans.*

9. Denote the hypotenuse by $2x$. Then the unknown leg $= x$. $\therefore 4x^2 - x^2 = 100$. $\therefore 2x = \dfrac{20\sqrt{3}}{3} = 11.547 +$. *Ans.*

10. Alt. $= \sqrt{36 - 9} = \sqrt{27} = 5.196 +$. *Ans.*

11. The altitude bisects the base (§ 117). Denote a side of the given $\triangle$ by $2x$. Then $4x^2 - x^2 = 8^2$ (§ 355). $\therefore 2x = \dfrac{16\sqrt{3}}{3} = 9.237 +$. *Ans.*

12. Denote a side of the square by x. Then $x^2 + x^2 = \overline{15}^2$. $\therefore x = \dfrac{15\sqrt{2}}{2} = 10.606 +$. *Ans.*

13. Denote the hypotenuse by x. Then $9 - x =$ the other leg. $\therefore x^2 - (9 - x)^2 = 9$. $\therefore x = 5, 9 - x = 4$. *Ans.*

14. See Ex. 21, p. 174. The radius of the small auxiliary $\odot$ is 6 in. $-$ 1 in. $=$ 5 in. $\therefore$ length of common tangent $= \sqrt{\overline{13}^2 - 5^2}$ in. $=$ 12 in. *Ans.*

15. See Ex. of § 410. Hence, $12 : 18 = x : 20 - x$. $\therefore x = 8, 20 - x = 12$. *Ans.*

16. See the figure of § 319. Let $AB = 6$, $BC = 8$. Then $AC = \sqrt{6^2 + 8^2} = 10$. Also $AF : AB = AB : AC$. $\therefore AF : 6 = 6 : 10$. $\therefore AF = 3.6$. *Ans.* Similarly, $BC = 6.4$. *Ans.* Also $AF : BF = BF : FC$ or $3.6 : BF = BF : 6.4$. $\therefore BF = 4.8$. *Ans.*

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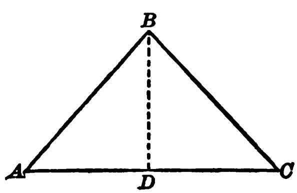
17.



Let ABC be the triangle and BD the altitude. Then $AB = 16$, $BC = 17$, $BD = 15$, $AD = \sqrt{\overline{16}^2 - \overline{15}^2} = \sqrt{31}$. Also $DC = \sqrt{\overline{17}^2 - \overline{15}^2} = 8$. $\therefore AC = 13.567+$. *Ans.*

18. Denote the segments of the hypotenuse by x and $10 - x$. Then $x : 4 = 4 : 10 - x$ (§ 319). $\therefore x = 2, 10 - x = 8$. The legs (by § 319) are $2\sqrt{5}$ or $4.472+$, and $4\sqrt{5}$ or $8.944+$. *Ans.*

19. $\dfrac{22 \times 28 \times 3000}{7 \times 5280 \times 12}$ mi. $= 4\frac{1}{6}$ mi. *Ans.*

20. $c = \dfrac{8800}{1400}$ yd. $d = \dfrac{c}{\pi} = \dfrac{8800 \times 7}{1400 \times 22}$ yd. $= 2$ yd. or 6 ft. *Ans.*

21. See § 385. $10.47 +$; $14.49 +$. *Ans.*

22. See § 385. Let $x =$ no. of degrees in the arc. $\therefore 14 = \dfrac{x}{180°} \times \frac{22}{7} \times 6$. $\therefore x = 133\frac{7}{11}°$. *Ans.*

23. See § 385. $12 \text{ in.} = \dfrac{90°}{180°} \times \dfrac{22r}{7}$. $\therefore 2r = 15\frac{3}{11}$ in. *Ans.*

24. $2\pi R - 2\pi r = 132 \text{ in.} - 88 \text{ in.}$ $\therefore \frac{44}{7}(R - r) = 44$ in. $\therefore R - r = 7$ in. *Ans.*

25. Velocity in miles per second $= \dfrac{44 \times 93{,}250{,}000}{7 \times 365\frac{1}{4} \times 24 \times 60 \times 60}$mi. $=$ $18.5+$ mi.

26. The diagonal of the square $= \sqrt{5^2 + 5^2} = 5\sqrt{2}$. Hence, the radius of the circumscribed circle $= \dfrac{5\sqrt{2}}{2} = 3.535+$. Hence, $c = \dfrac{110\sqrt{2}}{7} = 22.223+$. *Ans.*

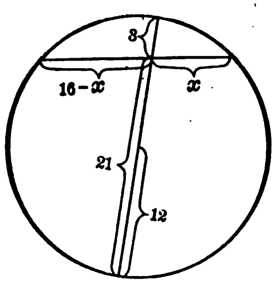
27. Diagonal of the rectangle $= \sqrt{12^2 + 5^2} = 13$. $\therefore r = \frac{13}{2}$. $c = 13\pi = 40.857+$. *Ans.*

28. Use § 385. Denote the central $\angle$ by x. Then $\dfrac{\pi r x}{180°} + 2r = \pi r$ or $\dfrac{\pi x}{180°} + 2 = \pi$. $\therefore x = 65\frac{5}{11}°$. *Ans.*

29. Denote a side of the square by x. $\therefore x^2 + x^2 = 100$ (§ 355). $\therefore x = 5\sqrt{2}$. $\therefore$ perimeter of square $= 20\sqrt{2}$. $\therefore 2\pi R = 20\sqrt{2}$. $\therefore \text{R} = 4.4997+$. *Ans.*

30. $x(14 - x) = 8 \times 3$(§ 323). $\therefore x = 12$ or 2. *Ans.*

31.



$x(16 - x) = 21 \times 3$ (§ 323). $\therefore x = 9, 7$. $16 - x = 7, 9$. *Ans.*

32. Denote the external segment of the second secant by x. Then $27x = 6 \times 24$ (§ 325). $\therefore\ x = 5\frac{1}{3}$. *Ans.*

33. The entire secant $= 9 + 16$ or 25. Let $x =$ length of the tangent. $\therefore\ 25 \times 9 = x^2$. $\therefore\ x = 15$. *Ans.*

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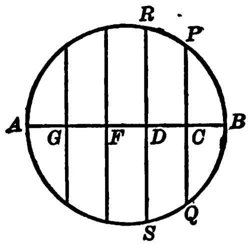
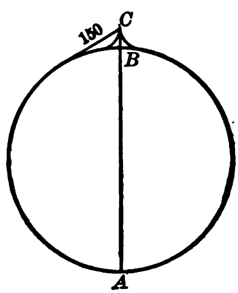
34. The line drawn from the point to the center ($= x$), the tangent, and the radius to the point of contact form a right △. $\therefore\ x^2 = \overline{24}^2 + \overline{18}^2$. $\therefore\ x = 30$. *Ans.*

35.



Let AB be the diameter and G, F, D, C the points of division. Then $AC = 48$, $CB = 12$. $\therefore\ PC = \sqrt{48 \times 12} = 24$ (§ 320), $PQ = 48$. Similarly, $RS = 24\sqrt{6}$ or $58.787+$, etc.

36.

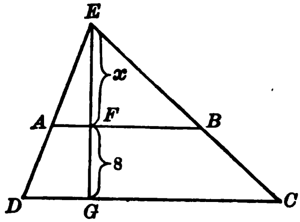


Let C be the top of the mountain. The limit of visibility is determined by the tangent drawn from the top of the mountain to the earth's surface. Denote the diameter of the earth by x. $\therefore\ AC = x + 3$. $\therefore\ 3(x + 3) = \overline{150}^2$ (§ 324). $\therefore\ x = 7497$ mi. *Ans.*

37. See Ex. 36. Denote the distance by x. But 100 ft. $= 0.019+$ mi. $\therefore\ x^2 = 0.019\ (8000.019)$ mi. $\therefore\ x = 12.309+$ mi. *Ans.*

38. The altitude of the upper $\triangle$ is 10. Hence, to find the base of the upper $\triangle$, $12 : 10 = 14$: line drawn (§ 316). $\therefore$ $11\frac{2}{3}$. *Ans.*

39.



Let $ABCD$ be the trapezoid; let the legs produced meet in E, and EG be $\perp$ DC. Denote EF by x. $\triangle$ AEB and DEC are similar (§ 304). $x : x + 8 = 12 : 20$ (§ 316). $\left.\begin{aligned} \therefore\ x &= 12. \\ x + 8 &= 20. \end{aligned}\right\}$. *Ans.*

40. See solution of Ex. 39. $\dfrac{b_1 h}{b_2 - b_1}, \dfrac{b_2 h}{b_2 - b_1}$. *Ans.*

41. See formula 4 for h_c, p. 294. $\therefore$ $\dfrac{21\sqrt{15}}{16}$ or $5.083 +$. *Ans.*

42. See formula 5 for m_c, p. 294. $\therefore$ $\frac{1}{2}\sqrt{106}$ or $5.147 +$. *Ans.*

43. See formula 6 for t_c, p. 294. $\therefore$ $\dfrac{21\sqrt{10}}{13}$ or $5.108 +$. *Ans.*

44. See formulas for h_c, m_c, t_c, p. 294. The medians are $\frac{1}{2}\sqrt{673}$ or $12.971 +$, $\frac{1}{2}\sqrt{592}$ or $12.165 +$, $\frac{1}{2}\sqrt{505}$ or $11.236 +$. The bisectors are $\dfrac{168\sqrt{5}}{29}$ or $12.953 +$, $\frac{3}{2}\sqrt{65}$ or $12.093 +$, $\dfrac{28\sqrt{13}}{9}$ or $11.217 +$. The altitudes are $12\frac{12}{13}$ or $12.923 +$, 12, 11.2. *Ans.*

1. No. acres $= \dfrac{300 \times 200}{2 \times 43560} = 0.688+$. *Ans.*

2. Use Formula 5 (Areas), p. 294. 84. *Ans.*

3. Use Formula 5 (Areas), p. 294. 10 sq. ch. $= 1\,A$. $203.33 + A$. *Ans.*

4. Draw the altitude. Then alt. $= \sqrt{34^2 - 8^2}$, etc. $264.36+$. *Ans.*

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5. Denote a side of the equilateral $\triangle$ by $2x$. Then $4x^2 - x^2 = 64$ (§ 355). $\therefore$ $2x = 9.236+$. Area $= 36.9504+$. *Ans.*

6. Denote a leg by x. $\therefore x^2 + x^2 = 144$ (§ 355). $\therefore x = 6\sqrt{2}$.
Area $= \dfrac{6\sqrt{2} \times 6\sqrt{2}}{2} = 36$. *Ans.*

7. Other leg $\sqrt{41^2 - 9^2} = 40$ (§ 356). $\therefore$ area $= 180$. *Ans.*

8. $\dfrac{b^2\sqrt{3}}{4} = 4\sqrt{3}$. $\therefore b = 4$. *Ans.*

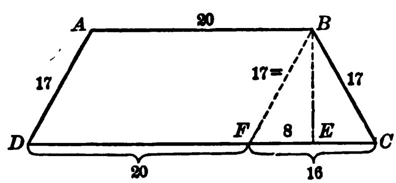
9. The no. of boards $= \dfrac{48 \times 24}{12 \times \frac{1}{2}} = 192$. *Ans.*

10. The no. of persons $= \dfrac{15 \times 9 \times 144}{27 \times 18} = 40$. *Ans.*

11. No. of acres $= \dfrac{90 \times 90}{43560} = 0.186-$. *Ans.*

12. See § 121. The diagonals divide the rhombus into four rt. $\triangle$, each of which has 17 for the hypotenuse and 15 for a leg. $\therefore \frac{1}{2}$ of the other diagonal $= \sqrt{17^2 - 15^2} = 8$. $\therefore$ area $= 240$. *Ans.*

13.



Let $ABCD$ be the trapezoid. Draw $BF \parallel AD$. $\therefore BFC$ is an isosceles $\triangle$ (§ 155). Draw its alt., BE. $\therefore BE = \sqrt{17^2 - 8^2} = 15$. $\therefore$ area $= 420$. *Ans.*

14. Denote the altitude of the upper $\triangle$ by x. $\therefore$ alt. of the trapezoid $= 18 - x$. Find, by use of similar $\triangle$, base of upper $\triangle = \dfrac{10x}{9}$. $\therefore$ area of trapezoid $= \left(\dfrac{5x}{9} + 10\right)(18 - x) = 80$. $\therefore x = 13.42-$. $\therefore$ upper base $= 14.907+$. *Ans.*

15. Draw lines from the center of the circle to the vertices of the polygon. In all the $\triangle$ thus formed the altitude is the same, viz.: the radius of the circle. $\therefore$ area of polygon $= \dfrac{20 \times 340}{2} = 3400$. *Ans.*

16. Denote the altitude by x. $\therefore (3x)\,x = 144$.

$$\left.\begin{aligned} \therefore\quad x &= 4\sqrt{3}, \text{ or } 6.928+. \\ 3x &= 12\sqrt{3}, \text{ or } 20.785+. \end{aligned}\right\} \textit{Ans.}$$

17. Denote a side of the hexagon by x. Then $\frac{6x^2\sqrt{3}}{4} = 200$. $\therefore x = 8.77+$ in. *Ans.*

18. By § 383, $c = 2\pi r$, or $p = 2\pi r$. $\therefore r = \frac{p}{2\pi}$. $\therefore K = \frac{\pi p^2}{4\pi^2} = \frac{p^2}{4\pi}$. (§ 390).

19. Denote the radius by x. $\therefore \pi x^2 = 81\pi + 1600\pi$. $\therefore x^2 = 1681$. $\therefore x = 41$ in. *Ans.*

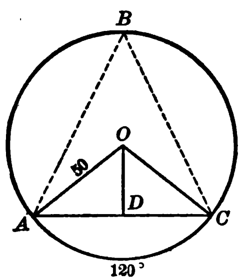
20. $\pi x^2 = \pi(\overline{20}^2 + \overline{28}^2 + \overline{29}^2)$. $\therefore x = 45$. *Ans.*

21. See § 394. $K = \frac{80^\circ}{360^\circ} \times \frac{22 \times \overline{50}^2}{7} = 1746+$. *Ans.*

22. Area of a sector of $60^\circ = \frac{60^\circ}{360^\circ} \times \frac{22 \times 2500}{7} = 1309.5238+$.

Area of an equilateral $\triangle$ of side $50 = \frac{\overline{50}^2\sqrt{3}}{4} = 1082.5319+$.

Area of a segment of $60^\circ = 226.99+$. *Ans.* Area of a segment of 300° = (area of the circle) − (segment of 60°) = $7630.15+$. *Ans.* A segment of 240° is cut off by the side of an inscribed equilateral $\triangle$ (segment AB). A side of $\triangle ABC$



$= 50\sqrt{3} = AC$. $\therefore OD = 25$. $\therefore$ segment $ABCD$ = sector $ABCO$ $+ \triangle AOC = \frac{1}{3} \times \frac{22}{7} \times 2500 + \frac{1}{2} \times 50\sqrt{3} \times 25 = 6320.627+$. *Ans.*

23. See § 394. Denote the angle by x. Then $\frac{x^\circ}{360^\circ} \times \frac{22}{7} \times 49 = 45$. $\therefore x = 105.2^\circ$. *Ans.*

24. Side of inscribed square $= 10\sqrt{2}$. Sum of the segments = (area of whole circle) − (area of inscribed square) $= 100\pi - 200 = 114.28+$. *Ans.*

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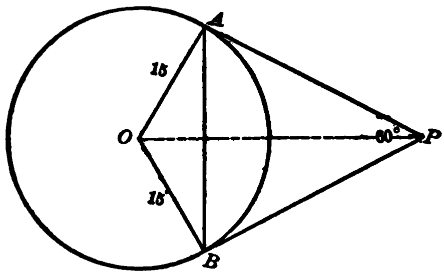
25. $\frac{1}{2}$ mi. $=$ 880 yd. $\therefore$ water area $= \frac{22}{4 \times 7}(\overline{880}^2 - \overline{100}^2) \times \frac{1}{4840}$ A.
$= 124.09 +$ A. *Ans.*

26.



Let O be the center of the circle and PA and PB tangents. $\triangle ABP$ is isosceles ($PA = PB$, by § 216). $\therefore \angle PAB = 60° = \angle PBA$ (§ 102). $\therefore \angle OAB = 90° - 60° = 30° = \angle OBA$. $\therefore \angle AOB = 120°$. $\therefore AB$ is a side of an inscribed equilateral $\triangle$. $\therefore AB = 15\sqrt{3}$. $\therefore$ area $OAPB = \triangle OAB + \triangle ABP = \frac{1}{2} \times 15 \times \frac{15\sqrt{3}}{2} + \frac{\overline{15}^2 \times 3\sqrt{3}}{4} = 225\sqrt{3} = 389.71+$. *Ans.*

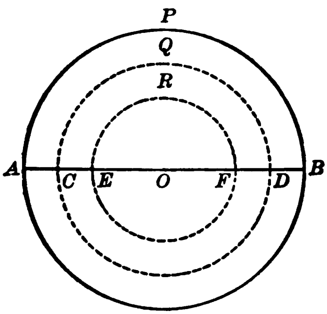
27. Denote the radius by x. $\therefore \frac{22}{7}x^2 = 43560$ sq. ft. $\therefore x = 117.7 +$ ft. *Ans.*

28. Join the centers of the three circles. The lines of centers pass through the points of tangency (§ 221) and form an equilateral $\triangle$. The included area is equal to an equilateral $\triangle$ whose side is $2r$, minus the sum of three sectors which together equal $\frac{1}{2}$ of one of the circles. $\therefore$ it $= \frac{(2r)^2\sqrt{3}}{4} - \frac{\pi r^2}{2} = r^2\left(\sqrt{3} - \frac{\pi}{2}\right)$.

29. The areas of the two similar $\triangle$s are to each other as the squares of any pair of corresponding altitudes (§§ 352, 316, 291). Denote the area of the entire $\triangle$ by K and of the upper by K'. $\therefore K : K' = \overline{18}^2 : 9^2 = 4 : 1$. But $K = 216$ sq. in. $\therefore K' = 54$ sq. in. *Ans.*

30. In this case K' must $= \frac{1}{2}K$. Denote the alt. of upper $\triangle$ by x. $\therefore K : \frac{1}{2}K = \overline{18}^2 : x^2$. $\therefore x = 9\sqrt{2}$ in., or $12.728-$ in. *Ans.*

31. Denote area of circle APB by $3K$. Denote area of circle ERF by K. Denote area of circle CQD by $2K$. $3K : K = \overline{30}^2 : \overline{EF}^2$



(§ 393). $\therefore$ $EF = 10\sqrt{3}$ in. $= 17.320+$ in., also $3K : 2K = \overline{30}^2 : \overline{CD}^2$. $\therefore$ $CD = 10\sqrt{6}$ in. $= 24.495-$ in. *Ans.*

1. Alt. $= \sqrt{10^2 - 8^2} = 6$. $\therefore$ area $= 48$. *Ans.*

2. Area of $\triangle = 468$ by formula 5 (Areas), p. 294. Denote radius of the circle by x. Then $\pi x^2 = 468$. $\therefore$ $x = 12.228+$. *Ans.*

3. $6\sqrt{3}$, or $10.392+$.

4. 14 in.; 16 in.; 19 in.; $7\sqrt{2}$ in.; $8\sqrt{2}$ in.; $9\sqrt{2}$ in. (or $9.899+$ in., $11.313+$ in., $12.727+$ in.). *Ans.*

5. See Ex. of § 410. 9, 12. *Ans.*

6. By drawing the diagonal which does not cut the $\angle$ of $60°$, the quadrilateral is divided into an equilateral and a rt. $\triangle$. $16.825+$. *Ans.*

7. Denote the diameter by x. Then $480 \times \pi x = 5280$ ft. $\therefore$ $x = 3.5$ ft. *Ans.*

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8. $\frac{h}{2}(12 + 16) = 112$ (§ 350). $\therefore$ $h = 8$. *Ans.*

9. Denote the radius of the circle by x. $\therefore$ $\pi x^2 = 100$ (§ 390). $\therefore$ $x = 5.6407+$. *Ans.*

10. $\pi x^2 = \dfrac{144\sqrt{3}}{4}$ (§ 390 and Ex. 4, p. 242). $\therefore$ $x = 4.4542-$. *Ans.*

11. $\pi x^2 = \frac{9}{2}(16 + 18)$ (§§ 390, 350). $\therefore$ $x = 6.977+$. *Ans.*

12. For area of circle see Ex. 18, p. 287. $K = \frac{\overline{12}^2}{4\pi}$ sq. yd. $= 11.45+$ sq. yd. *Ans.* Area of square $= 3^2$ sq. yd. $= 9$ sq. yd. *Ans.* Area of eq. $\triangle = \frac{4^2\sqrt{3}}{3}$ sq. yd. $= 6.9282+$ sq. yd. *Ans.*

13. $400 : 125 = 360° : \angle$ of sector. $\therefore$ $\angle$ of sector $= 112\frac{1}{2}°$. *Ans.*

14. Let x = width of field included in ft. $6x$ = perimeter of running track in ft. $2x^2$ = area included by track in sq. ft. But $6x = 2640$ ft. $\therefore$ $x = 440$ ft. $\therefore$ $2x^2 = \frac{440 \times 880}{43560}$ A. $= 8.8+$ A. *Ans.*

15. $r = \frac{6 \text{ in.}}{2} = 3$ in. $R = \sqrt{3^2 + 3^2}$ in. $= 3\sqrt{2}$ in.

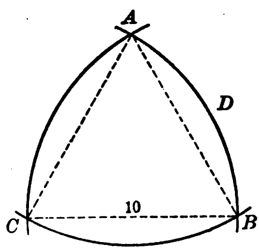
By § 390, $\left.\begin{array}{l} \pi r^2 = 9\pi \text{ sq. in.} = 28.285 + \text{ sq. in.} \\ \pi R^2 = 18\pi \text{ sq. in.} = 56.571 + \text{ sq. in.} \end{array}\right\}$ *Ans.*

16. Denote the other leg by x. Then $x + 8$ = hypotenuse. $\therefore$ $(x + 8)^2 = x^2 + \overline{12}^2$ (§ 355). $\therefore$ $x = 5$. $\therefore$ area $= 30$. *Ans.*

17. Let x = a leg of the $\triangle$. $\therefore$ $x^2 + x^2 = \overline{20}^2$ (§ 355). $\therefore$ $x = 10\sqrt{2}$. Area $= \frac{1}{2}(10\sqrt{2})^2 = 100$. *Ans.*

18. Use § 319. 16, $4\sqrt{5}$, $8\sqrt{5}$ (or 16, 8.944+, 17.888+). *Ans.*

19.



Let ABC be the given figure. Draw the chords of these arcs. These chords form an equilateral $\triangle$ whose side is 10 (§ 363). Area of $\triangle ABC = \frac{\overline{10}^2\sqrt{3}}{4} = 43.301+$. Area segment ABD = sector $ACBD - \triangle ABC = \frac{\pi\overline{10}^2}{6} - 43.301+ = 9.07+$. $\therefore$ required area $= \triangle ABC + 3$ segment $ADB = 70.5403+$. *Ans.*

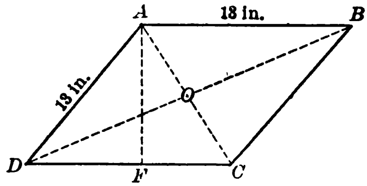
20. $\sqrt{43560} = 208.7-$. $\therefore$ perimeter of square field $= 834.841+$ ft. For circle $43560 = \pi R^2$. $\therefore$ $R = \sqrt{\frac{43560}{\pi}}$. $\therefore$ $C = 2\pi\sqrt{\frac{43560}{\pi}} =$

$2\sqrt{\pi}\sqrt{43560}$ ft. $= 740.016+$ ft. $\therefore$ difference $= (834.841 - 740.016+)$ ft. $= 94.83+$ ft. *Ans.*

21. Area is doubled (§ 345); is doubled (§ 345); quadrupled (§ 346).

22. Denote the side of the $\triangle$ by x; then $\frac{x^2\sqrt{3}}{4} = \frac{100\pi}{4}$. $\therefore x = 13.47-$. *Ans.*

23.



Let $ABCD$ be the given rhombus and AF its altitude. Then $AF \times 13$ in. $= 156$ sq. in. $\therefore AF = 12$ in. $\therefore DF = \sqrt{13^2 - 12^2}$ in. $= 5$ in. $\therefore FC = 13$ in. $- 5$ in. $= 8$ in. In the rt. $\triangle AFC$, $AC = \sqrt{12^2 + 8^2}$ in. $= 4\sqrt{13}$ in. $= 14.422+$ in. *Ans.* Area of rhombus $= 156$ sq. in. $= \frac{AC \times DB}{2} = 2\sqrt{13}$ in. $\times DB$. $\therefore DB = \frac{156 \text{ in.}}{2\sqrt{13}} = 6\sqrt{13}$ in. $= 21.633+$ in. *Ans.*

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1. Alt. $= 8$ dm. $\therefore K = 64$ sq. dm. (§ 343). *Ans.*
2. The sides are 6 m., 7 m., 8 m. $\therefore$ by Formula 5 (Areas), p. 294, $K = \frac{21}{4}\sqrt{15}$ sq. m. $= 20.333+$ sq. m. *Ans.*
3. $R = 1.4$ m. $\therefore K = \pi\overline{1.4}^2$ sq. m. $= 6.16+$ sq. m. *Ans.*
4. The other leg $= \sqrt{17^2 - 15^2}$ dm. $= 8$ dm. $\therefore K = 60$ sq. m. *Ans.*
5. See Ex. 18, p. 287. 7.9545 + sq. dm. *Ans.*
6. $K = \pi\overline{100}^2$ sq. m. $= \frac{10000}{10000}\pi$. Ha. $= \pi$. Ha. $= 3.1428+$ Ha. *Ans.* 3.14 Ha. $= (3.14+)(2.471+)$A. $= 7.7659+$ A. *Ans.*
7. The side $= 8$ dm. $\therefore$ other side $= \sqrt{35^2 - 8^2}$ dm. $= 34.07+$ dm. Area $= 2.7258+$ sq. m. $= 4225.09+$ sq. in. *Ans.*
8. The bases are 6 m. and 2 m., and the alt. is 8 m. $\therefore K = 32$ sq. m. *Ans.*
9. The dimensions are 70 m. and 200 m. $\therefore K = 1.4$ Ha. $= 3.459+$ A. *Ans.*

10. 9 dm. and 4 dm. *Ans.*

11. Side of square $= 18$ in. $\therefore$ area of square $= 324$ sq. in. $\therefore \pi R^2 = 324$ sq. in. $\therefore R = \sqrt{\frac{324}{\pi}}$ in. $\therefore R$ in dm. $= \frac{10}{39.37}\sqrt{\frac{324}{\pi}}$ dm. $= 2.578 +$ dm. *Ans.*

12. Circf. of wheel in meters $= \frac{10000}{5000}$ m. $= 2$ m. $\therefore$ diameter of wheel $= \frac{2}{\pi}$ m. Diameter of wheel in feet $= \frac{2 \times 39.37}{\pi \times 12} = 2.087+$ ft. *Ans.*

1. (Group 86). Use figure of Ex. 8, p. 97 (text-book). Denote the given adj. $\angle$s by $2x$ and $2y$. Then $x + y = 90°$ (Hyp.). $\therefore 2x + 2y = 180°$ (Ax. 4).

2. Use § 80.

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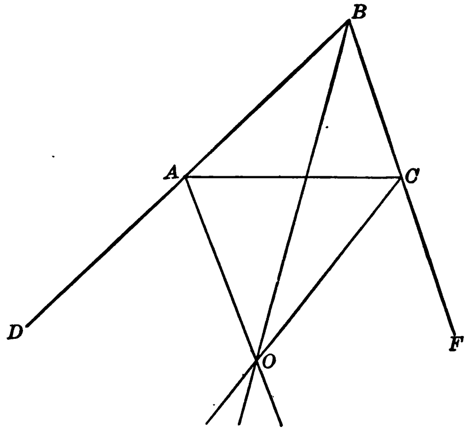
3. With the center of the given $\odot$ as a center and the given distance as a radius, describe a circle. From the given point draw a tangent to this circle. (§ 264), etc.

4.



Let $\angle ABO = \angle OBC$, and $\angle ACO = \angle OCF$. It is required to show that $\angle BOC = \frac{1}{2} \angle BAC$. Let $p = \angle ABO = \angle OBC$, $q = \angle BAC$, $r = \angle BCA$, $x = \angle BOC$. Then in $\triangle BOC$, $x + p + r + \frac{1}{2} \angle ACF = 180°$ (§ 102). Also in $\triangle ABC$, $2p + q + r = 180°$ (§ 102). But $\angle ACF = 2p + q$ (§ 103). $\therefore \frac{1}{2} \angle ACF = p + \frac{1}{2} q$ (Ax. 5). By Axs. 1 and 9, $x + p + r + p + \frac{1}{2} q = 2p + q + r$. $\therefore x = \frac{1}{2} q$ (Ax. 3).

5. Radius of inscribed ⊙ = 9 in., of circumsc. ⊙ = $9\sqrt{2}$ in., etc.

6. Denote $\widehat{DB}$ by x, and $\widehat{BC}$ by y. Then $74° = \frac{1}{2}(x + 108°)$ (§ 240). $\therefore x = 40°$. Similarly, $106° = \frac{1}{2}(y + 112°)$. $\therefore y = 100°$. $\angle ADB = 104°$ (§ 235), etc.

7. A straight line which is the ⊥ bisector of the line of centers. Use §§ 221, 123.

8. Use Ax. 5, §§ 82, 79, 23, 24.

9. Two pairs. See figure of § 322. (Draw chords DB and AC.) ∴ △ DFB and AFC are mutally equiangular (§ 235) and similar (§ 304). Likewise △ DFA and BFC are similar.

10. Area of trapezoid = 44 sq. in. Area of square = 64 sq. in. To find per cent wasted, $\frac{60}{168} = .357+$. ∴ 35.7 + per cent. *Ans.*

11. Draw AF. $\triangle ABF = \triangle AFE$ (§ 117). $\therefore BF = FE$. Then prove $FE = EC$, by showing $\angle ECF = 45° = \angle EFC$ (§§ 82, 102, 114).

12. Draw the radius to the point of contact. Thus a rt. △ is formed (§ 210) whose hypotenuse is 24 cm. and one leg of which is 12 cm. $\sqrt{\overline{24}^2 - \overline{12}^2}$ cm. = 20.78+ cm. *Ans.*

13. 36°. By § 168, each ∠ of a regular pentagon = 108°, etc.

14. Let the diagonals intersect at O. Then $OP = OR$ (Ax. 3). $OQ = OS$ (Ax. 3), etc.

15. See Ex. 7, p. 284. 3 in. *Ans.*

16. Produce PD to meet the circle at F. Then $\widehat{BF} = \widehat{BP}$ (§ 202). $\angle APB \stackrel{m}{=} \frac{1}{2}\widehat{BP}$ (§ 241), $\angle BPF \stackrel{m}{=} \widehat{BF}$ (§ 235), etc.

17. Reduces to § 255.

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18. Denote a leg of the given rt. △ by x. Then $\frac{x^2}{2} = 1296$ sq. in. (§ 343). $\therefore x = 50.91 +$ in. *Ans.*

19. Let h = altitude of the given △. Then h is a constant (§ 343). Hence, the locus is two lines || base of given △ and at the distance h from this base.

20. In the △ ACF and DCB, $DC = AC$ (sides of same square), $CF = CB$ (same reason), $\angle ACF = \angle BCD$ (each $= \angle ACB + 1$ rt. ∠), etc. Use §79.

21. Let $ABCD$ be the given quadrilateral and BP and PC the bisectors of the ∠ ABC, BCD. Denote each half of $\angle ABC$ by a,

and each half of $\angle BCD$ by b. Then $2\angle a + 2\angle b + \angle D + \angle A = 360°$ (§ 167). Also $\angle a + \angle b + \angle P = 180°$ (§ 102). $\therefore 2\angle a + 2\angle b + \angle D + \angle A = 2\angle a + 2\angle b + 2\angle P$, etc.

22. Let AP and QB intersect in R outside the given circle.

$\therefore \angle R \stackrel{m}{=} \frac{1}{2}(\overset{\frown}{AQ} - \overset{\frown}{BP})$ (§ 242).

$\stackrel{m}{=} \frac{1}{2}(\overset{\frown}{AQ} + \overset{\frown}{AP} - \overset{\frown}{AP} - \overset{\frown}{BP})$.

$\stackrel{m}{=} \frac{1}{2}(180° - \overset{\frown}{AB}) =$ a constant.

Hence, if on the chord AB a segment of a circle is constructed which shall contain $\angle R$ (§ 265), the arc of the segment is the locus of R.

In like manner if AP and BQ intersect inside the circle it may be shown that $\angle ARB \stackrel{m}{=} \frac{1}{2}(180° + \overset{\frown}{AB})$.

23. Arc of $22\frac{1}{2}° = \frac{3\pi}{2}$; but $22\frac{1}{2}° = \frac{1}{16}(360°)$. $\therefore$ arc of $360° = 16\left(\frac{3\pi}{2}\right) = 24\pi$.

Hence, $2\pi r = 24\pi$. $\therefore r = 12$. $\therefore \pi r^2 = 452\frac{4}{7}$. Hence, $452\frac{4}{7}$ ft. *Ans.*

24. Denote the radii of the given circles by x and y. Then $\pi x^2 + \pi y^2 = 20$ sq. yd. $\pi x^2 - \pi y^2 = 15$ sq. yd., etc. $\therefore x = 2.359 +$ yd. $y = 0.891 +$ yd. *Ans.*

25. This problem reduces to one of constructing a right $\triangle$, given the hypotenuse (viz.: a side of the required square) and the sum of the legs (viz.: a side of the given square). See Ex. 14, p. 170.

26. Draw the diagonals of the $\square$ and from their point of intersection draw a $\perp$ to same line. Then the sum of the $\perp$s from either pair of vertices of the $\square$ = twice the $\perp$ from the point of intersection of the diagonals (Ex. 7, p. 213).

27. If the centers of the two circles are joined with each other and also with the points where the two circles intersect, two equilateral triangles are formed in each of which a side is r, the radius of each circle.

Hence, sum of areas of the two $\triangle$s $= \frac{r^2\sqrt{3}}{2}$ (Ex. 4, p. 242). The remainder of the area common to the two circles is four segments of 60° each in a circle with radius r. Area of one segment = sector of $60° -$ one of the $\triangle$s $= \frac{\pi r^2}{6} - \frac{r^2\sqrt{3}}{4} = \frac{2\pi r^2 - 3r^2\sqrt{3}}{12}$.

Hence, area common to the two circles

$$= \frac{r^2\sqrt{3}}{2} + \frac{4(2\pi r^2 - 3r^2\sqrt{3})}{12} = \frac{4\pi r^2 - 3r^2\sqrt{3}}{6}. \quad \textit{Ans.}$$

28. $\pi r^2 = 80. \quad \therefore r = \sqrt{\frac{280}{11}}.$

Arc of $80° = (\frac{2}{9}) \frac{44}{7} \sqrt{\frac{280}{11}} = \frac{8}{63}\sqrt{280 \times 11} = 7.04+.$ *Ans.*

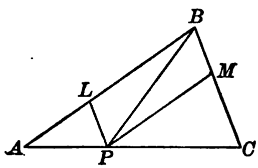
29. See Ex. 3, p. 241. Denote the radius of the inscribed circle by r.

$\therefore K = \frac{1}{2}(a + b + c)r.$

$$\therefore r = \frac{2K}{a+b+c} = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a+b+c},$$

where $s = \frac{1}{2}(a + b + c)$. (See formula 5 under areas, p. 294.)

30.



Let $PLBM$ be the resulting $\square$. $\therefore AL : LB = AP : PC = 2 : 3$ (§ 293). $\therefore \triangle ABP = \frac{2}{5} \triangle ABC$ (§ 345). $\therefore \triangle LBP = \frac{3}{5} \triangle ABP = \frac{6}{25} \triangle ABC$. $\square LBMP = 2 \triangle LBP = \frac{12}{25} \triangle ABC$. Hence, 12 : 25. *Ans.*

31. Let F be the midpoint of BE. From D draw $DH \parallel BE$ and meeting AB at H. Let K be the midpoint of HD. Then the locus is the broken line $CFKA$.

PAGE 293

32. Add $\triangle ECD$ to each of the equivalent $\triangle$s. $\therefore \triangle$s BCD and ACD are equivalent. Since these $\triangle$s have the same base, CD, they must have equal altitudes, etc.

33. Let x be the exact diameter in inches of the required pipe. Then $2^2 : x^2 = 2 : 3$. $\therefore x = \sqrt{6} = 2.76 +$ (§ 393). $\therefore$ 3 in. pipe. *Ans.*

PAGE 297

1. Denote the projection by x, then by §§ 102, 355, $x^2 + x^2 = 10^2$.
$\therefore x = 5\sqrt{2}$. *Ans.*

2. Use an equilateral $\triangle$ whose side is 10. The altitude bisects the base. $\therefore$ projection $= 5$. *Ans.*

3. $\frac{1}{2}a$. See Ex. 2.

PAGE 298

1. $2\sqrt{31}$ (or 11.135+). For $\angle BCD = 60°$. $\therefore CD = 5$ (see Ex. 3, p. 297).

2. $\frac{5}{2}$. For $\overline{20}^2 = \overline{14}^2 + \overline{12}^2 + 2 \times 12 \times CD$, etc.

3. $\frac{1}{3}$.

4. Produce PR through R to T making $RT = \frac{1}{3}$ of 300 yd., or $= 100$ yd. Also produce QR through R to S making $RS = \frac{1}{3}$ of 219 yd. or $= 73$ yd. Measure TS. Then $PQ = 3\ TS$.

PAGE 299

1. $2\sqrt{31}$ (or 11.135+). For $DC = 5$ (see Ex. 3, p. 297). $\therefore \overline{AB}^2 = \overline{10}^2 + \overline{12}^2 - 2 \times 12 \times 5 = 124$.

PAGE 300

2. Obtuse (see § 417).
3. Acute (see § 418).
4. Obtuse. Acute. Right. Acute.

---

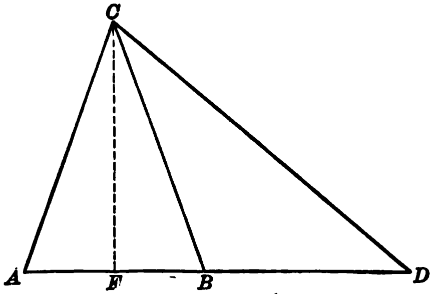
1. $\angle P = 54° - 27° = 27°$. $\therefore BP = AB = 3\frac{1}{4}$ mi. *Ans.*

2. Use § 417 and find $\overline{AP}^2 = (\frac{13}{4})^2 + (\frac{13}{4})^2 + 2\,(\frac{13}{4})\frac{13}{8}$.
$\therefore AP = \frac{13}{4}\sqrt{3}$ mi. $= 5.629 +$ mi. *Ans.*

3. Through S draw a straight line, TSP, along the bank of the stream so that $\angle RTS = \frac{1}{2} \angle RSP$. Measure TS. Then $RS = TS$.

4. Denote the point taken in the base by P. Draw the altitude BQ. Then, $\overline{BC}^2 = \overline{BP}^2 + \overline{PC}^2 - 2\,PQ \times PC$ (§ 418) $= \overline{BP}^2 + (PC - 2\,PQ)PC = \overline{BP}^2 + AP \times PC$.

5.



Draw $CF \perp AB$. Then $\overline{CD}^2 = \overline{BC}^2 + \overline{BD}^2 + 2\,BD \times FB$. But $AF = FB$ (§ 118). $\therefore 2\,BF = AB$. $\therefore \overline{CD}^2 = \overline{BC}^2 + \overline{BD}^2 + AB \times BD$ (Ax. 9) $= \overline{BC}^2 + BD\,(BD + AB) = \overline{BC}^2 + BD \times AD$.

6. Denote the hypotenuse by a, and the last leg by x. Then $\sqrt{a^2 - x^2}$ is the required leg, $a : \sqrt{a^2 - x^2} = \sqrt{a^2 - x^2} : x$. $\therefore a^2 - x^2 = ax$ (§ 278), etc. Use § 327.

7. By § 417, $\overline{AC}^2 = \overline{10}^2 + \overline{20}^2 + 2\,(20)\,5\sqrt{2} = 782.84+$.

$$\therefore AC = 27.97 + \text{ ft. } \quad \textit{Ans.}$$

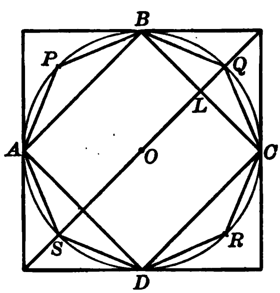
By § 418, $\overline{BD}^2 = \overline{10}^2 + \overline{20}^2 - 2\,(20)\,5\sqrt{2} = 217.16-$.

$$\therefore BD = 14.73 + \text{ ft. } \quad \textit{Ans.}$$

8. Quote theorem of § 417.

PAGE 301

1.



$OL = \frac{1}{2}\sqrt{2}$. $\therefore LQ = 1 - \frac{1}{2}\sqrt{2}$. $\therefore$ by § 320

$\overline{BQ}^2 = SQ \times LQ = 2\,(1 - \frac{1}{2}\sqrt{2})$.

$\therefore BQ = .7653+$.

$\therefore$ perimeter of the inscribed octagon $= 6.1229+$ in. *Ans.*

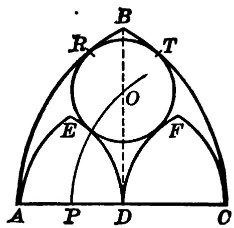
PAGE 303

1. In the shape of a circle. 2. $40A$. $46.8 + A$. $50.9 + A$.

3. $ARPQ$ is a $\square$ (Def.) $\therefore \triangle QAR = \triangle QPR$ (§ 156). $RC \parallel QP$, hence $\triangle PRC : \triangle QPR = RC : QP$ (§ 345). $\triangle QPR : \triangle BQP = PR : BQ$ (§ 345). But $\triangle$ BQP and PRC are similar (§§ 112, 304). $\therefore RC : QP = RP : BQ$ (§ 312). $\therefore \triangle PRC : \triangle QPR = \triangle QPR : \triangle BQP$ (Ax. 1). But $\triangle QPR = \triangle QAR$, etc.

PAGE 306

1.



The line BD is the locus of the centers of all circles tangent to both AB and BC. For in § 221, if the points A and B come together as a point at which the two circles are tangent to each other, this point will be on OO'. Hence, in Ex. 1, if the circle O is tangent to AB and BC, $AO = AT - OT$. Also $CO = RC - RO$. $\therefore AO = CO$. $\therefore O$ is on BD (§ 120).

Also if P is the midpoint of AD, an arc described with C as a center and CP as radius is the locus of all circles tangent to both AB and DF (for every point in it is equidistant from the two arcs) (§ 188, Axs. 3, 5), etc.

2. BD. No. AED, DFC, RBT, the circle O. The circle O.

3. Denote the base of the given rectangle by b, its altitude by h, the base of the required rectangle by b'. Then find a fourth proportional to b', b, and h (§ 297). This will be altitude of the required rectangle, etc.

PAGE 307

6. Point of intersection of the diagonals of the square. Point of intersection of the diagonals of the rectangle. Center of the circumscribed circle. Center of the circle.

7. The centroid of the triangle (see Ex. 5, p. 306). Center of the circle circumscribed about the pentagon.

8. At the centroid of the triangle.

9. Use §§ 112, 304.

10. Use §§ 114, 304.

11. Use §§ 97, 304, etc.

13. Draw ⊥s from two opposite vertices of the parallelogram to the diagonal joining the other two vertices. Apply §§ 417, 418 to the four △s into which the parallelogram is divided, etc.

PAGE 308

17. Use §§ 95, 295.

19. Denote the base of the given △ by b and its altitude by h. Construct the mean proportional between b and $\frac{1}{2}h$ (§ 321). Construct a square which has this mean proportional as a side.

PAGE 309

23. Construct a △ equivalent to the given pentagon (see Ex. 21). Then use Ex. 19, p. 308.

25. In Ex. 2, change m into 2, and n into 3, etc.

27. Construct a regular pentagon (as by drawing the chords FD, DB, etc. in the diagram of Ex. 26) and draw its diagonals.

28. First construct a regular hexagon. Then draw its diagonals.

29. .00025 + %.

30. .04 + %.

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31. Hence, construct a line which is a fourth proportional to 16, 19, and s. Then with H as a center and this line as a radius describe an arc intersecting BD, etc.

KEY TO SOLID GEOMETRY

PAGE 312

4. $\sqrt{3}$.

PAGE 313

8. Use § 160.
9. 12 ft. $12\sqrt{3}$ ft. or 20.78 + ft.
10. Pyramid.
11. Prism.
12. Prism.
13. Cylinder, on frustum of a cone.
14. Cone.
15. Sphere.

PAGE 315

2. $_4C_3 = \dfrac{4 \times 3 \times 2}{1 \times 2 \times 3} = 4.$ *Ans.* $_4C_2 = \dfrac{4 \times 3}{1 \times 2} = 6.$ *Ans.*
3. 3.
4. $_4C_2 = \dfrac{4 \times 3}{1 \times 2} = 6.$ *Ans.*
5. $1\frac{1}{3}$ times.
6. 3. For the intersecting lines, and each line taken with the point, determine a plane.
7. Yes. No.
8. No.

PAGE 316

9. Place a straight edge of the plank on the tennis court in many different positions and directions and observe whether all points of the straight edge are in contact with the surface of the court in every position of the plank (§ 429).
10. See Ex. 9.
11. Because three points determine a plane (§ 430). Four legs are often used to obtain greater weight-supporting power, greater stability, etc.
12. So that the plane of the instrument may be readily changed.

PAGE 317

10. Cube and octahedron.

11. Cube.

12. Cube and octahedron.

13. Cube. Octahedron.

14. Three (§ 430). So that the wagon body shall be able to sustain a greater weight and shall not be so easily overturned.

15. One point. See § 430.

PAGE 321

2. ∡ *PBA* and *PBC* (§ 440). By nailing a board to the post *PB* and to the sill *BC* at points at a considerable distance from *B*.
By the spirit level (the form called the "plumb and level" and described in Ex. 3).

PAGE 322

3. To two sides.

4. If the pole is cylindrical, apply the spirit level to two sides of the pole, a quadrant's distance apart, and adjust the pole till the spirit level at each of the places is in a vertical position (§ 440).
If the pole is noticeably conical, apply the spirit level to two diametrically opposite sides of the pole and adjust the pole till the deviations of the level from the vertical are equal in the two positions. Then in like manner apply the spirit level to two other diametrically opposite sides of the pole.

5. A plane $\perp$ the axis (§ 441).

7. The cube is easier to make, and cubes also pack together better.

PAGE 323

Ex. Use § 432. A plane is determined by two parallel lines.

PAGE 325

1. 10 each. For ${}_5C_3 = \dfrac{5 \times 4 \times 3}{1 \times 2 \times 3} = 10$, and ${}_5C_2 = \dfrac{5 \times 4}{1 \times 2} = 10$.

2. 2. **4.** $\triangle ABC$ is a rt. $\triangle$. $\therefore AC = \sqrt{8^2 + 6^2} = 10$. $\therefore FC = 10$. *Ans.*

5. If *A* is the point and *MN* the plane (see figure, p. 324) locate three points, *D*, *C*, *F*, in *MN*, which are equidistant from *A*. Through *D*, *C*, *F*, construct a circle with center *B*. Draw *AB*.

6. Three. § 430.

PAGE 329

2. Hold the pencil parallel to the floor or wall (§ 452).

3. Use § 79.

4. See Ex. 5, p. 325.

5. $AE = EC$ (Hyp.). $AO = OC$ (§ 159). $\therefore EO \perp AC$ (§ 121). Similarly, $EO \perp BD$. $\therefore EO \perp$ plane $ABCD$ (§ 440). $\therefore AO = BO = OC = OD$ (§ 446).

6. Prove $\triangle ABC = \triangle BCF$ (§ 79). $\therefore AC = CF$, etc.

7. Use § 447.

8. $BO = OC$, and $BD = DC$. Use § 121.

9. $\therefore AB = a_1$, $BF = \frac{1}{2}a$ $\therefore AF = \sqrt{a^2 - (\frac{1}{2}a)^2} = \dfrac{a\sqrt{3}}{2}$. *Ans.*

$FO = \frac{1}{3} FD$ (Ex. 9, p. 253) $= \frac{1}{3} AF = \dfrac{a\sqrt{3}}{6}$.

$\therefore AO = \sqrt{\overline{AF}^2 - \overline{FO}^2} = \sqrt{\dfrac{3a^2}{4} - \dfrac{a^2}{12}} = \sqrt{\dfrac{2a^2}{3}} = \dfrac{a}{3}\sqrt{6}$. *Ans.*

PAGE 330

10. A plane surface midway between the walls and parallel to them.

11. A straight line $\perp$ plane of the $\triangle$, and passing through the center of its circumscribed $\odot$.

12. A circle whose center is in the base of the $\triangle$, and whose radius = altitude of the $\triangle$.

13. The given $\triangle$ is a rt. $\triangle$ (§ 419). The locus is a line $\perp$ plane of the $\triangle$ and passing through the midpoint of AC.

PAGE 331

Ex. The locus is a circle lying in the plane which bisects AB and is $\perp$ it. The center of the circle is the point where AB intersects the plane, and the radius of the circle is $\sqrt{q^2 - \frac{1}{4}p^2}$.

PAGE 332

Ex. A circle on the floor with its center vertically beneath P, and with a radius = 9 ft.
The same with a radius = $\sqrt{b^2 - a^2}$.

PAGE 335

1. $2 : 7 = 3 : CD$. $\therefore CD = 10.5$. *Ans.*

2. $4\frac{2}{7}$, $5\frac{5}{7}$. $GD = 8\frac{4}{7}$.

3. Use §§ 456, 461.

4. No.

5. Draw BD and AC. By §§ 300, 304 show that line joining midpoints of AD and $DC = \frac{1}{2}AC$ and $\parallel AC$. Similarly, in $\triangle ABC$ show that the line joining the midpoints of AB and $CB = \frac{1}{2}AC$ and $\parallel AC$. $\therefore$ the quadrilateral is a $\triangle$ by *Ax*.1 and §§ 464, 161.

6. Use § 460. Or use and §§ 441, 455.

7. § 464.

8. Use §§ 452, 436, 100, 440, 455.

9. §§ 456, 146.

10. Prove $BQ = BC$, then $AQ = AC$, etc.

11. Mark off the equal lines FP and FQ. Prove $AQ = AP$, $BQ = BP$, etc.

12. The locus is a plane $\parallel$ the given plane, and bisecting the $\perp$ from P to the given plane.

PAGE 339

Ex. See §§ 63, 66, 112, 114, etc.

PAGE 341

1. If we know that one plane is $\perp$ another, we are saved the labor of proving that a line drawn in one of the planes $\perp$ this line of intersection, is also $\perp$ the other plane.

2. C–AB–Q, Q–AB–S, S–AB–P, P–AB–C, R–CS–N, R–CS–M, A–BQ–C, A–BQ–S.

3. 1 : 5; 1 : 1.

PAGE 343

1. § 440.

2. Plane $ABCD \perp$ planes OBC and ODC (§ 482). $\therefore OC \perp$ plane $ABCD$ (§ 488).

3. §§ 488, 463.

4. $AB \perp MN$. $\therefore$ plane $ABC \perp MN$ (§ 482). $AC \perp RS$. $\therefore$ plane $ABC \perp$ plane RS (§ 482). Plane $ABC \perp$ line RN (§488). $\therefore BC \perp RN$ (§ 436).

5. See Ex. 4.

6. See Ex. 8, p. 329.

PAGE 344

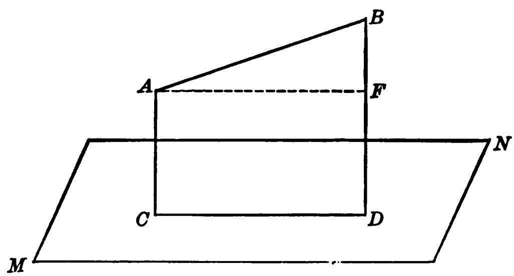
7. $AK = KB$ (Hyp.). Prove $AP = BP$. $\therefore KP \perp AB$ (§ 121). $\therefore \triangle QKP = \triangle PLS$, etc.

PAGE 345

1. Seven, two. In the plane angle fewer lines are to be made, watched, and reasoned about.
2. §§ 462, 482.
3. The locus is a circle lying in the plane which bisects the line BC at right angles. The center of the circle is the foot of a $\perp$ from A to the plane, and the radius of the circle is 8 in.
4. The locus is the straight line formed by the intersection of the planes which bisect the line BC and the line PQ at right $\angle$s.
5. Use the plane $\angle$s of the dihedral $\angle$s.
6. Through the point in the given line draw a $\perp$ to the given planes (§ 444), and then form the projections of the line in the planes. Prove that these projections are parallel (§ 456). Use § 97, etc.
7. Let the distance between the two given points be denoted by $2a$. Also denote the distance from the third point by b. Then if $b > a$, the locus is a circle; if $b = a$, the locus is a point. If $b < a$, the locus does not exist since there is no point which fulfills the required conditions.

PAGE 348

1. $180° - 2\,(47°) = 86°$. *Ans.* 2. $120°$; $90°$.
3.



Given the plane MN, AB a straight line oblique to MN, and CD the projection of AB on MN.
To prove $AB > CD$.
Proof. Draw $AF \perp BD$. Then in rt. $\triangle ABF$, $AB > AF$ (§ 135). But $AF = CD$ (§ 155). $\therefore AB > CD$ (Ax. 9).

4. Use figure p. 324. Let $AB \perp MN$, $AD = AC$. To prove $\angle ADB = \angle ACB$. In $\triangle$s ACB and ABD, $AD = AC$ (Hyp.); $AB = AB$ (Ident.). $\therefore \triangle ACB = \triangle ADB$ (§ 117). $\therefore \angle ADB = \angle ACB$ (corr. $\angle$s of $= \triangle$s).

5. From the given point draw a $\perp$ the given plane (§ 445). Through this $\perp$ pass a plane in any direction (§ 434). The plane last drawn is the plane required. For it is $\perp$ given plane (§ 482). Hence the problem has an infinite number of solutions.

6. Construct the plane $\angle$ of the dihedral $\angle$ and bisect the plane $\angle$. Use § 480.

7. From the point where the three lines meet, mark off three equal segments on the three given lines. Through the three terminal parts of these segments pass a plane. From the point where three lines meet draw a $\perp$ to the plane (§ 445). Use § 447, etc.

8. Construct a plane and line $\perp$ it as in Ex. 7. Through the point where the lines meet draw a plane $\perp$ the $\perp$ (§ 443), etc.

9. Use §§ 492, 443.

10. $5\sqrt{3}$ or $8.66+$ in.; $5\sqrt{2}$ or $7.07+$ in.; 10 in.; 0.

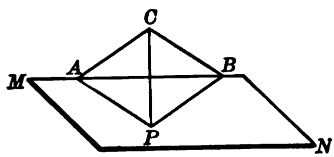
11. The supplement of its inclination to the plane. For in the diagram p. 347, produce CB to meet the circle at Q. Then $AQ > AP$, since $CQ > CP$. Use § 138.

12. § 120, 122; §§ 126, 127.

13. A plane midway between the two planes and parallel to them.

14. A plane which bisects the dihedral $\angle$ formed by the walls.

15.



Let ABC be the isosceles $\triangle$ with $AC = CB$, and MN a plane through AB, P the projection of C on MN. Then $\triangle ACP = \triangle CBP$ (§ 117). $\therefore \angle CAP = \angle CBP$.

PAGE 351

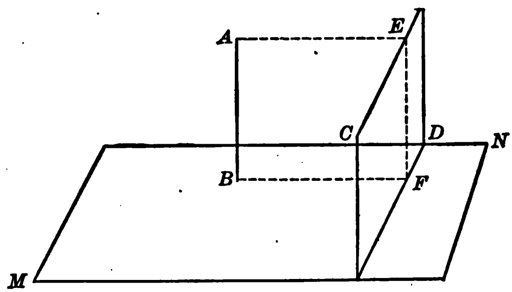
1. Denote any point on BR produced by X. $R\text{–}PQB$, $R\text{–}PTB$, $R\text{–}PQX$, $R\text{–}PTX$; trirectangular trihedral; birectangular trihedral.

2. Yes. Use §§ 488, 444. 4. Four. 180°. 5. Six. 240°.

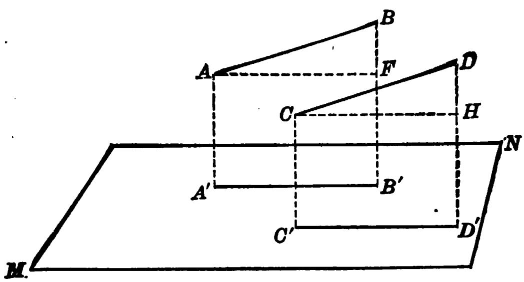
6. Trihedral, birectangular, trirectangular, isosceles.

7. The planes intersect in two parallel lines (§ 456). Draw a plane $\perp$ these $\parallel$ lines. Use §§ 456, 97, 480.

PAGE 355

1. Use §§ 458, 455.
2. If they are not ||, the given line and plane will meet in some point. Draw a ⊥ from this point to the line of intersection of the two planes. This ⊥ is ⊥ second plane (§ 485). Hence we shall have two ⊥s drawn from a given point to a given plane, which is impossible (§ 445). Hence the given line and plane cannot meet, and are ∴ ||.
3. Pass a plane through *PA* and *PB* and another plane through *PC* and *PD*. These planes are || (§ 460). ∴ *AB* || *CD* (§ 456). *PA* = *PC* (§ 457), etc.
4. Two. One.
5. 

Given *AB* ⊥ plane *MN*; *AB* || plane *CD*.
To prove *CD* ⊥ *MN*.
Proof. Pass a plane through *AB* to intersect *CD* in *EF*. Then *AB* || *EF* (§ 452). ∴ *EF* ⊥ *MN* (§ 462). ∴ *CD* ⊥ *MN* (§ 482).
CONVERSELY. If *CD* and *AB* ⊥ *MN*, *CD* || *AB*. See Ex. 2.

6. 

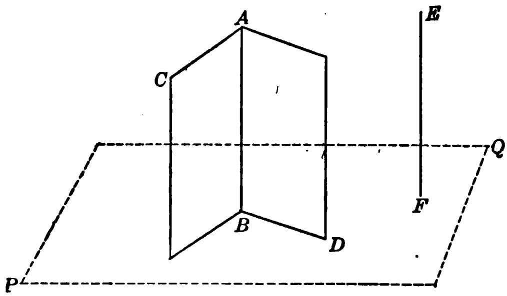
Given *AB* and *CD* two || and = lines, and *A′B′* and *C′D′* their respective projections on the plane *MN*.

To prove $A'B' \parallel$ and $= C'D'$.

Proof. AA' and CC' are $\perp$ MN (§ 479). $\therefore AA' \parallel CC'$ (§ 463). $AB \parallel CD$ (Hyp.). $\therefore$ plane $BAA' \parallel$ plane DCC' (§ 460). $\therefore A'B' \parallel C'D'$ (§ 456).

Draw $AF \perp BB'$ and $CH \perp HD'$. In rt. $\triangle ABF$ and CDH, $AB = CD$ (Hyp.); $\angle ABF = \angle CDH$ (§ 461). $\therefore \triangle ABF = \triangle CDH$ (§ 117). $\therefore AF = CH$, etc.

7.



Given $EF \parallel$ planes BC and AD.

To prove $EF \parallel AB$, the intersection of BC and AD.

Proof. Construct the plane $PQ \perp EF$ (§ 442). Then plane $BC \perp PQ$ (Ex. 5); and plane $AD \perp PQ$ (Ex. 5). $\therefore AB \perp PQ$ (§ 488). $\therefore EF \parallel AB$ (§ 463).

9. See Exs. 4 and 5, p. 343.

10. $DE \perp$ plane CEA (§ 440). $\therefore$ plane $DEA \perp$ plane CEA (§ 482). But the plane $DCA \perp$ plane DEA (Hyp.). $\therefore$ both plane CAD and plane CAE are $\perp$ plane DAE. $\therefore CA \perp$ plane DAE (§ 488).

PAGE 356

11. On the figure to Ex. 15, p. 348, take a point R midway between A and B and draw RC and RP. Then $\angle CRP$ is the plane $\angle$ of the dihedral $\angle P\text{–}AB\text{–}C$ (§ 121).

$\therefore RC = \sqrt{8^2 - 3^2} = \sqrt{55}$. $\therefore 2\,\overline{CP}^2 = 55$.

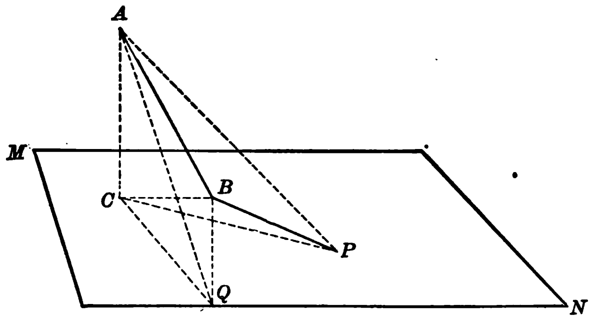
$\therefore CP = 5.24 +$ in. *Ans.*

12. See Ex. 2, p. 355.

13. See the figure of Ex. 6, p. 355. The lines themselves are not necessarily parallel.

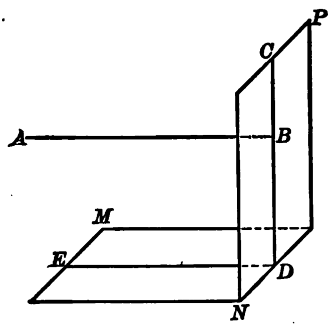
For in the plane DCC' a line could be drawn through C oblique to CD, having the projection $C'D'$, but not $\parallel AB$.

14.



In the plane MN draw $BQ \perp BC$ and $= BP$. Then $CP > CQ$ (§ 138). $\therefore AP > AQ$ (§ 447). $\therefore \angle ABP > \angle ABQ$ (§ 138). But $\angle ABQ =$ rt. $\angle$ (Ex. 10, p. 335). $\therefore \angle ABP$ is obtuse.

15.



Given line $AB \perp$ plane PN at B, $AB \parallel$ plane MN, MN and PN intersecting in ND.

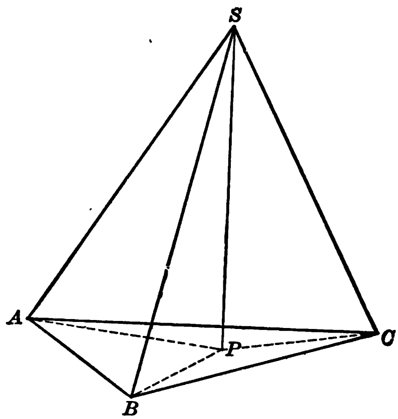
To prove $PD \perp MN$.

Proof. Pass a plane through $AB \perp ND$, intersecting MN in ED and PN in CBD. Then $AB \perp BD$ (§ 436); $AB \parallel ED$ (§ 452). $\therefore ED \perp BD$ (§ 100). $ND \perp ED$ and BD (§ 436), etc.

17. This follows from Ex. 5, p. 335.

18. Let S–ABC be the trihedral $\angle$. Let plane SAP bisect the dihedral $\angle B$–AS–C, and plane SBP bisect the dihedral A–SB–C. Let the planes SAP and SBP intersect in the line SP. Through SP and SC pass a plane. It can be shown that plane SPC bisects $\angle B$–SC–A. For SP, being in the plane SAP, every point in SP

is equidistant from the planes SAC and SAB (§ 493). Similarly, since SP is in the plane SBP, every point is equidistant from the planes SBA and SBC (§ 493). ∴ every point in SP is equidistant



from the planes SAC and SBC (Ax. 1). ∴ plane SPC bisects the dihedral $\angle B\text{–}SC\text{–}A$ (§ 493). Hence the three planes which bisect the dihedral ∠s of $S\text{–}ABC$ meet in the line SP.

19. Rt. $\triangle QOR$ = rt. $\triangle QOS$ (§ 110). ∴ $QR = QS$ (corr. sides of = △s). ∴ rt. $\triangle PQR$ = rt. $\triangle PQS$ (§ 79). ∴ $PR = PS$ (corr. sides of = △s). $PR \perp OR$, and $PS \perp OS$ (Ex. 10, p. 335).

20. **Prove △s** AOP, BOP, COP equal (§ 83). ∴ $\angle APO = \angle BPO = \angle CPO$, etc. Use § 110.

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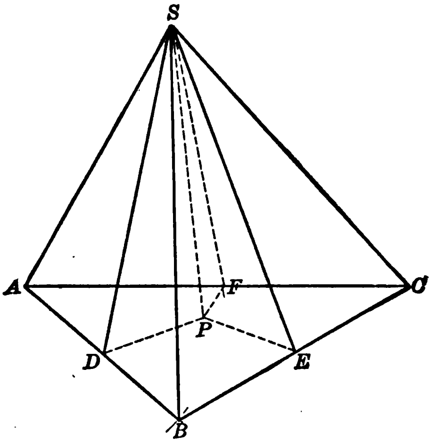
22. It is necessary first to prove the converse of Ex. 21. That is, in the figure of Ex. 19, given $PR \perp OR$, $PS \perp OS$, and $PR = PS$, to prove that P lies in the plane which bisects $\angle SOR$ and is ⊥ its plane. **Proof.** Draw $PQ \perp$ plane ROS. Then in the rt. △s PQR and PQS, $PQ = PQ$ (Ident.), $PR = PS$ (Hyp.). ∴ $\triangle PQR = \triangle PQS$ (§ 117). Also $QR \perp OR$ and $QS \perp OS$ (Ex. 11, p. 335). ∴ OQ bisects $\angle SOR$ (§ 126). Plane $POQ \perp$ plane ROS, for it contains the line $PQ \perp ROS$ (§ 482). ∴ P lies in the plane required.

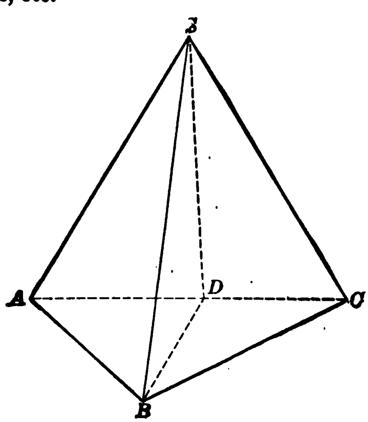
Let $S\text{–}ABC$ be a trihedral ∠. Let plane SDP bisect $\angle ASB$ and be ⊥ its plane. Let plane SEP bisect $\angle BSC$ and be ⊥ its plane.

Let planes SDP and SEP intersect in the line SP. Let SF be a line in the plane ASC bisecting the $\angle\ ASC$. Then, since SP is in plane SDP, every point in SP is equidistant from the lines SA and SB (Ex. 21). Also, since SP lies in the plane SPE, every point in



SP is equidistant from SB and SC (Ex. 21). $\therefore$ every point in SP is equidistant from the lines SA and SC (Ax. 1). $\therefore$ SP lies in the plane which bisects $\angle\ ASC$ and is $\perp$ its plane (converse of Ex. 21). Hence the three planes which bisect the face $\angle$s of S–ABC at right $\angle$s meet in a line, etc.

23.



In the trihedral $\angle S\text{–}ABC$, let the face $\angle ASB$ = face $\angle BSC$.
To prove dihedral $\angle B\text{–}AS\text{–}C = \angle B\text{–}SC\text{–}A$.

Proof. In the face $\angle ASC$ draw SD bisecting the $\angle ASC$. Then in the trihedral $\measuredangle$ $S\text{–}ABD$ and $S\text{–}BDC$, $\angle BSD = \angle BSD$ (Ident.). $\angle ASB = \angle BSC$ (Hyp.). $\angle ASD = \angle CSD$ (Constr.). $\therefore$ the corresponding dihedral $\measuredangle$ of $S\text{–}ABD$ and $S\text{–}BCD$ are equal (§ 509). $\therefore \angle B\text{–}AS\text{–}C = \angle B\text{–}SC\text{–}D$.

24. $\angle BSC + \angle DSC > \angle BSD$ (§ 507). Also $\angle ASD = \angle ASD$ (Ident.). Add by Ineq. Ax. 1, etc.

25. $AB \parallel ab$ and $BC \parallel bc$ (§ 456). $\therefore \angle ABC = \angle abc$ (§ 461). $\therefore PB : Pb = AB : ab$ (§ 302). Also $pb : pB = bc : BC$ (§ 302). $\therefore AB : ab = bc : BC$ (Ax. 1). $\therefore AB \times BC = ab \times bc$ (§ 278). $\therefore \triangle ABC \approx \triangle abc$ (Ex. 9, p. 242).

26. The parts of the trihedral $\measuredangle$ may be taken as equal and in the same order. $\therefore$ the trihedral $\measuredangle$ may be made to coincide.

27. Draw SO and $S'O'$. $\therefore S\text{–}OAB = S'\text{–}O'A'B'$ (Ex. 26). $S\text{–}OAC = S'\text{–}O'A'C'$ (Ex. 26). $S\text{–}OBC = S'\text{–}O'B'C'$ (Ex. 26). Adding $S\text{–}ABC = S'\text{–}A'B'C'$ (Ax. 3). (Subtract one pair from sum of the other two if O falls outside $\triangle ABC$).

---

1. A circle in the plane which bisects AB at right angles. The center of the circle is the midpoint of AB, and its radius is $\sqrt{39}$ in.

2. A straight line $\parallel$ both planes, at a distance d from M, and e from N.

3. A plane passing through the given point and $\parallel$ the given plane.

4. A straight line which lies in the plane which bisects at right angles the line joining the two given points. Also the locus line is $\parallel$ the two given planes and midway between them.

5. A straight line which is the intersection of the plane which bisects the dihedral $\angle$ formed by the two given planes and of the plane which is the $\perp$ bisector of the line joining the two given points.

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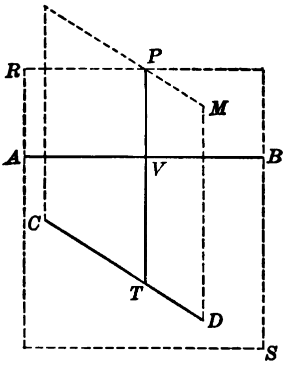
6. Two straight lines formed by the intersection of the plane which bisects the dihedral angle formed by the given intersecting planes, and two other planes each $\parallel$ the first given plane, and at the given distance from it.

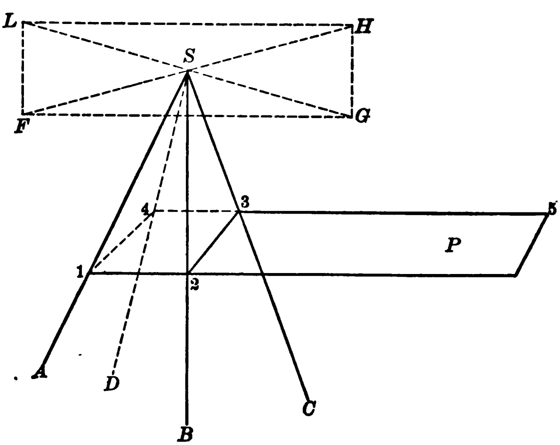
8. Pass a plane through the three given points, and in that plane draw a circle through the three given points. At the center of this circle construct a $\perp$ to the plane containing the three points.

9.



Let P be the given point and AB and CD the given lines. Through P and AB pass the plane RS (§ 434). Through P and CD pass the plane CM (§ 434). Then CM will intersect RS in the required line PT. For the points P, V, T are in both planes and hence must be in their line of intersection (§ 439).

10.



Let S–$ABCD$ be the given tetrahedral ∠ and P the given point. Produce the opposite faces ASB and DSC to meet in the line LG. Also produce planes ASD and BSC to meet in the line FH (§ 439). Through LG and FH pass a plane $LFGH$ (§ 434). Through P pass

a plane 15 || plane *FHLG* (§ 459) intersecting the faces *ASB* in 12, *BSC* in 23, *CSD* in 34, *ASD* in 14. Then 1234 is the required ▱. For since 15 || *LFGH*, *LG* || 12 (§ 456). Similarly, *LG* || 43. ∴ 12 || 43 (§ 464.) Also *FH* || 14 and 23. Hence 14 || 23 (§ 464). ∴ 1234 a ▱ by definition of ▱.

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1. § 464.
2. §§ 436, 437, 439 (1), 442, 443, 444, 445, 449.
3. §§ 432, 440, 441, 446, 447, 451, 453, 462, 463, 494.
4. §§ 439 (2), 452, 455, 458, 482, 485, 486, 490.
5. §§ 457, 485, 486.
6. §§ 467, 476, 493.
7. §§ 476, 480.
8. §§ 500, 501, 507.
9. § 509.
10. §§ 499, 500, 503, 504, 505.
11. Arrange the points in a straight line.
12. Yes. No.
13. One. One if the two lines are not parallel; otherwise an infinite number.

PAGE 363

1. Four.
2. Rectangular.
3. Right parallelopipeds. The same.
4. 60 sq. in. 76 sq. in.
5. 4.5 sq. in. 5.36 + sq. in.
7. 1344 sq. ft.
8. $= \sqrt{5^2 + 4^2 + 2^2} = \sqrt{45}$. ∴ 6.708 + in. *Ans.*

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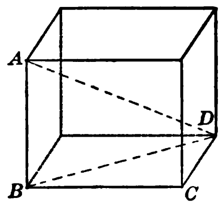
1. 112 sq. in.
2. $8\frac{5}{8}$ sq. ft.
3. 886.95 sq. in.
4. $65\frac{1}{8}$ sq. ft.
5. 34.6 ft.
6. Area of base $= \frac{16\sqrt{3}}{4}$ sq. in. $= 4\sqrt{3}$ sq. in. ∴ $S = 180$ sq. in., $T = 193.856+$ sq. in. *Ans.*
7. Area of base = 90. ∴ $S = 1080$, $T = 1260$. *Ans.*
8. K of hexagon $= \frac{6 \times 8^2\sqrt{3}}{4}$ sq. ft. ∴ $S = 960$ sq. ft., $T = 1292.5+$ sq. ft. *Ans.*
9. $T = 832$ sq. in., $T = 2\,(pq + pr + qr)$ sq. ft. *Ans.*
10. $6 \times \overline{15}^2$ sq. in. = 1350 sq. in. *Ans.*

11. $3\frac{1}{9}$ sq. ft.

12. Denote the edge by x. Then $6x^2 = 1296$ sq. in. $\therefore x = 6\sqrt{6}$ in. $= 14.696 +$ in. *Ans.*

13.



Let AD be the cube with edge $= 5$ in. Draw the diagonal of the base, BD. Then $AB \perp$ plane BCD (§ 519). $\therefore \angle ABD$ is a rt. $\angle$ (§ 436). Also $\triangle BCD$ is a rt. $\triangle$ (def. of a square). Then, $\overline{AD}^2 = \overline{AB}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 = 3 \times 5^2 = 75$. $\therefore AD = 5\sqrt{3}$ in. $= 8.660+$ in. *Ans.*

14. See Ex. 13. Denote the edge of the cube by x. Then $\overline{12}^2 = 3x^2$. $\therefore x = \sqrt{48}$ ft. $T = 6x^2 = 288$ sq. ft. *Ans.*

15. §§ 436, 149.

16. § 451.

17. §§ 515 (1), 159.

18. See diagram and method of solution in Ex. 13.

19. 3480 sq. ft.

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21. Two, viz.: a side of the base and the altitude.

22. Total area $= 978.5$ sq. ft.

$\frac{978 - 100}{32} = 27.4+$. $\therefore 28$. *Ans.*

23. $64x + 240 = 816$. $\therefore x = 9$. $\therefore 9$ ft. *Ans.*

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2. $12\,(3\sqrt{2}) = 36\sqrt{2}$. $\therefore 50.91+$ sq. in. *Ans.*

3. $12\,(3\sqrt{3}) = 36\sqrt{3}$. $\therefore 62.35+$ sq. in. *Ans*

4. 36 sq. in.

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3. 3375 cu. in.

4. 216.

5. 560.

6. 48 cu. yd. Multiply the three dimensions of the room.

PAGE 371

7. $15\frac{5}{8}$ cu. in.

8. $11\frac{25}{64}$ cu. in.

PAGE 373

Ex. 2 : 5.

PAGE 374

1. 5 : 12.
2. The latter.
3. 9 : 1.
4. 3 : 16. 160 lb.

PAGE 375

1. 3375 cu. in. 1.57+ bu.
2. One edge.
3. Three.

PAGE 376

4. Denote an edge of the cube by x.
Then $6x^2 = 450$ sq. in. $\therefore x = 5\sqrt{3}$ in.
$V = 649.518+$ cu. in.
5. 12.
6. $6\frac{2}{9}$ cu. ft. $74\frac{2}{3}$ lumber ft.
7. 20.

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Ex. 3900 lb.

PAGE 378

1. 432 cu. in.
2. 643.96+. 629.66+.
3. 12.6 sq. in.

PAGE 380

1. $226\frac{5}{8}$ cu. ft.
2. 321.408 cu. in.
3. 294 cu. ft.
4. 112.1 cu. ft.
5. 11.5 in.
6. 173.2+ cu. in.
7. 1497+.
8. 450 cu. ft.
9. $2338\frac{1}{3}$ cu. ft.
10. 1440.
11. $133\frac{1}{3}$.
12. Denote the number of feet in the depth of the cistern by x. Then
$$12 \times 8 \times x \text{ cu. ft.} = \frac{10000 \times 231}{1728} \text{ cu. ft.} \quad \therefore x = 14- \text{ ft. } \textit{Ans.}$$
13. Denote the number of inches in a side of the base by x. Then
$$2x(x^2) = \frac{2150.42}{4}. \quad \therefore x = 6.45+ \text{ in. } \textit{Ans.} \quad 2x = 12.90+ \text{ in. } \textit{Ans.}$$
14. 4725 lb.

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16. 15,771+ lb.

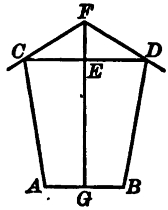
17. $293\frac{1}{3}$ cu. yd. \$63.07.

18. $13\frac{17}{18}$ tons

19. 90.

20. $\text{No. min.} = \dfrac{2 \times 43{,}560 \times 144}{18 \times 10 \times 24 \times 60} = 48\frac{2}{5}.$ *Ans.*

21.



Measure *AD*, *CD*, *EG*, *EF*. Also the length of the crib.

PAGE 384

1. § 447.

2. § 562 and Ex. 7, p. 213.

3. 1024. 1280.

4. 191.73 sq. ft.

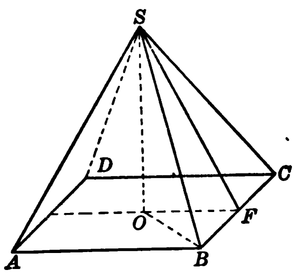
5. $13\frac{13}{16}$ sq. ft.

6. $11\frac{71}{128}$ sq. in.

7. $11\frac{7}{8}$ sq. ft.

8. \$291.60.

9.



$OF = 8$, $SF = 17$. $\therefore SO = \sqrt{\overline{17}^2 - 8^2}$. $\therefore SO = 15$. *Ans.*

10. In the figure to Ex. 9, $SO = 15$, $SB = 17$, to find BC. $\angle OFB$ is a rt. $\angle$. Denote an edge of the base by $2x$. Then $BF = OF = x$. In the rt. $\triangle SOB$, $\overline{SB}^2 = \overline{OB}^2 + \overline{SO}^2 = \overline{BF}^2 + \overline{OF}^2 + \overline{SO}^2$. $\therefore \overline{17}^2 = x^2 + x^2 + 225$. $\therefore x = 4\sqrt{2}$. $\therefore 2x = 8\sqrt{2} = 11.313+$. *Ans.*

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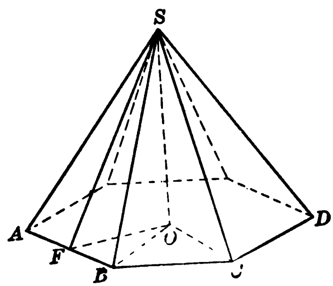
11. In the figure to Ex. 9, $SC = 25$, $BC = 14$, $FC = 7$, to find OS. In the rt.$\triangle$ SFC, $\overline{SC}^2 = \overline{SF}^2 + \overline{FC}^2$. $\therefore 625 = \overline{SF}^2 + 49$. $\therefore SF = 24$. In the rt. $\triangle SOF$, $OF = 7$. $\therefore \overline{OS}^2 = \overline{SF}^2 - \overline{OF}^2 = 576 - 49 = 527$. $\therefore OS = 22.9+$. *Ans.*

12. See Ex. 9, p. 329. $h = 2\sqrt{6} = 4.898+$.

13. See figure of Ex. 9. $SO = 15$, $AB = 16$, $OF = 8$. $\therefore SF = \sqrt{15^2 + 8^2} = \sqrt{289} = 17$. $\therefore S = 544$, $T = 800$. *Ans.*

14. $S = 185.1+$, $T = 228.4+$.

15.

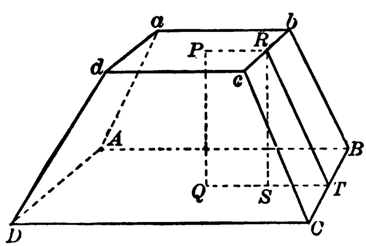


In the regular hexagonal pyramid $S\text{-}ABCD$, with slant height SF, let $AB = 4$, $SO = 21$, then $BO = 4$, $OF = 2\sqrt{3}$, $SF = \sqrt{21^2 + (2\sqrt{3})^2}$, etc. Then $S = 255.39+$, $T = 296.96+$. *Ans.*

16. In the figure of Ex. 9, let $SB = 25$, $SF = 24$. Then find $BF = 7$. $\therefore BC = 14$. Hence $S = 672$, $T = 868$. *Ans.*

17. $S = 12\sqrt{3} = 20.78+$, $T = 16\sqrt{3} = 27.71+$.

18.



Let aC be the frustum of a regular square pyramid, PQ its altitude, and RT its slant height. Let $dc = 6$, $DC = 18$, $PQ = 8$.

Draw QT, and PR, and $RS \perp QT$. Then $RS = PQ = 8$. $ST = QT - PR = 9 - 3 = 6$. $\therefore RT = \sqrt{8^2 + 6^2} = 10$. $\therefore$ lateral area $= \frac{1}{2}(72 + 24)\ 10 = 480$. *Ans.*

19. 11.5872. **20.** $111\frac{11}{12}$ sq. ft. **21.** 45°.

22. Prove $\triangle FOB = \triangle DOH$. $\therefore OF = OH$.
$\therefore \triangle FOP = \triangle POH$.

23. $PQ \parallel P'Q'$ (§ 456). $\therefore$ Prove $PQRS \sim P'Q'R'S'$.
Prove $P'Q' = 2\ PQ$. Use § 353.

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Ex. Denote area of section by x.

$\therefore x : b^2 = (a - c)^2 : a^2$. $\therefore x = \dfrac{b^2\,(a - c)^2}{a^2}$. *Ans.*

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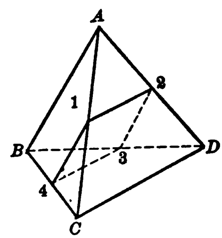
1.



Let A–BCD be the pyramid ,and 1234 the section $\parallel AB$ and CD. Then, since $CD \parallel 1234$, $CD \parallel 12$ (§ 452). In like manner, $CD \parallel 43$, $\therefore 12 \parallel 43$ (§ 464). Similarly, $14 \parallel 23$, since each is $\parallel AB$. $\therefore 1234$ is a ▱ (def.).
A rectangle; for pass a plane through $AB \perp CD$. Then 12 and 34 $\perp$ this plane (§ 462), and 14 and 23 are $\parallel$ it (§ 451), etc.

2. See Ex. 9, p. 253.

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Ex. $225\sqrt{3}$ or 389.71+ sq. ft.

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1. 323.826 cu. ft. **2.** 13,826 cu. in. **3.** $\dfrac{3l}{m}$ sq. ft.

4. $b = \dfrac{64\sqrt{3}}{4}$ sq. in. $\therefore$ by § 569, $V = 110.8+$ cu. in. *Ans.*

5. In the base the other leg $= \sqrt{\overline{29}^2 - \overline{21}^2} = 20$. $\therefore b = 210$. $\therefore V = 1400$. *Ans.*

6. For figure, see Ex. 9, p. 384. $V = 31.74+$. *Ans.*

7. Hence on figure for Ex. 9, p. 384, $\angle SFO = 45°$. $\therefore \angle OSF = 45°$. $\therefore SO = OF$. But $OF = 5$. $\therefore SO = 5$. $\therefore V = 166\frac{2}{3}$. *Ans.*

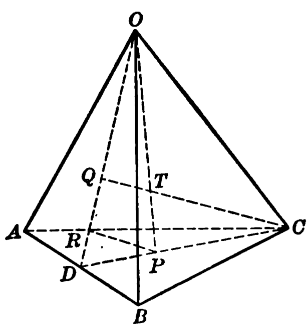
8. \$2,619,434.67–. *Ans.*

9. Find the volume of the spire as if solid; then find the volume of the hollow interior, and subtract the latter from the former. $V = 6690\sqrt{3}$ cu. ft. $= 11587.5+$ cu. ft. *Ans.*

10. 81 ft.

11. Denote the edge of the cube by a. Then the altitude of the given pyramids is $\frac{1}{2}a$. Hence its volume $= \frac{1}{3}a^2 \times \frac{a}{2} = \frac{1}{6}a^3$, etc.

12.



Given the regular tetrahedron $O\text{–}ABC$, with CD the altitude of the base, and OP the altitude of the tetrahedron. Draw OD. Then $OD \perp AB$ (Ex. 10, p. 335). Draw CQ and $PR \perp OD$. $AB \perp OD$ and DC (constr.). $\therefore AB \perp$ plane DOC (§ 440). $\therefore$ plane $OAB \perp$ plane DOC (§ 482). $\therefore RP$ and $QC \perp$ plane AOB (§ 485). $\therefore \triangle RPD$ similar $\triangle CQD$ (§ 305). $DC = 3\,DP$ (Ex. 9, p. 253.) $\therefore QC = 3\,RP$ (§ 302.)

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13. Denote the four $\perp$s by x, y, z, v, the area of a face of the tetrahedron by K, and the altitude of the tetrahedron by h. Then volume of tetrahedron $= \frac{1}{3}K \times h$ (§ 568). Pass a plane through the given point and each edge of the given tetrahedron. These planes will

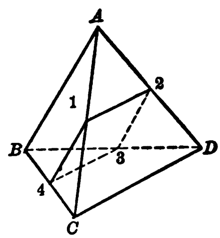
divide the given tetrahedron into four small tetrahedrons each with a base equal to K and with altitudes x, y, z, v, respectively. Then sum of small tetrahedrons $= \frac{1}{3}K(x+y+z+v)$ (§ 568). $\therefore$ $\frac{1}{3}K(x+y+z+v) = \frac{1}{3}K \times h$ (Ax. 1), or $x+y+z+v=h$ (Ax. 5).

14. $\frac{1}{6}$ volume of parallelopiped. **15.** $\frac{1}{6}$.

16. See Ex. 15. Hence, $\frac{a^3}{6}$ = volume.

17. See the figure of Ex. 12, p. 391. In the rt. $\triangle$ CDB, $\overline{BC}^2 - \overline{DB}^2 = \overline{CD}^2$. $a^2 - (\frac{1}{2}a)^2 = \overline{CD}^2$. $\therefore CD = \frac{a}{2}\sqrt{3}$. $\therefore PC = \frac{a}{3}\sqrt{3}$. In rt. $\triangle$ OPC, $OP = \sqrt{a^2 - \left(\frac{a}{3}\sqrt{3}\right)^2} = \sqrt{a^2 - \frac{a^2}{3}} = \frac{a\sqrt{6}}{3}$. Area of $ABC = \frac{a^2\sqrt{3}}{4}$. $\therefore$ volume $= \frac{1}{3}\left(\frac{a^2\sqrt{3}}{4}\right)\left(\frac{a\sqrt{6}}{3}\right) = \frac{a^3\sqrt{2}}{12}$.

18.



Let 1, 2, 3, 4 each be the midpoint of the edge in which it lies. Then in $\triangle$ ACD, 12 $\parallel$ CD (§§ 300, 304). In $\triangle$ BCD, 43 $\parallel$ CD (§§ 300, 304); also 14 $\parallel$ AB, 23 $\parallel$ AB. $\therefore$ 12 $\parallel$ 43, and 14 $\parallel$ 23 (§ 464). The lines lie in one plane (§ 432). $\therefore$ they form a ▱ (§ 161).

19. They are the diagonals of a ▱. Use § 159.

20. In the figure to Ex. 18, let F be the midpoint of CD and draw $F1$, $F2$, $F3$, $F4$. Since $AB = CD$, $12 = 23 = 34 = 41$ (§§ 300, 304 and Ax. 1). $\triangle$ $F12$ is equilateral. In like manner (and by Ax. 1), $\triangle$s $F34$, $F23$, $F14$ are equilateral, etc.

21. The octahedron is composed of two regular square pyramids, the height of each of which is $2\sqrt{2}$ in.

$\therefore V = \frac{64\sqrt{2}}{3}$ cu. in. $= 30.169+$ cu. in. *Ans.*

22. Altitude of the pyramid $= 5\sqrt{2}$ in.

$\therefore V = 100\sqrt{2}$ cu. in. $= 141.42+$ cu. in. *Ans.*

23. Pass planes through the axis and each lateral edge. Find the sum of the volumes of the small pyramids thus formed.

24. Diagonal of the base $= x\sqrt{2}$.

$$\therefore h = \sqrt{m^2 - \frac{x^2}{2}}, \text{ etc.}$$

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25. 15.27+ ft.

26. Area of each of the two triangular faces

$$= \tfrac{1}{2}\,(20)\left(\frac{20\sqrt{3}}{3}\right) \text{ sq. ft.} = \frac{200\sqrt{3}}{3} \text{ sq. ft.}$$

Area of each of the two trapezoids

$$= \tfrac{1}{2}\,(30+10)\frac{20\sqrt{3}}{3} \text{ sq. ft.} = \frac{400\sqrt{3}}{3} \text{ sq. ft.}$$

$$\therefore \text{ total area of the roof } = \frac{1200\sqrt{3}}{3} \text{ sq. ft.} = 692.82+ \text{ sq.ft.} \quad \textit{Ans.}$$

PAGE 395

1. 74 cu. in.
2. 70.896 cu. ft.
3. $103\frac{97}{162}$ cu. in.
4. 2000.18+.

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5. See § 573. $b_1 = \frac{25}{16}$ sq. ft., $b_2 = \frac{9}{16}$ sq. ft. $\therefore b_1b_2 = \frac{15}{16}$ sq. ft. $V = \frac{14}{3}\left(\frac{25}{16} + \frac{9}{16} + \frac{15}{16}\right)$ cu. ft. $= 171.5$ lumber ft. *Ans.* The mid-section of the frustum is a square with each side $\frac{9+15}{2}$ in. $= 12$ in. $\therefore$ volume of prism $= 14$ cu. ft. $= 168$ lumber ft. Difference $= 3.5$ ft. *Ans.*

6. Denote area of section by x. Then $x : 24 = 4^2 : 6^2$. $\therefore x = \frac{32}{3}$ sq. ft. $\therefore 14\frac{2}{9}$ cu. ft., $33\frac{7}{9}$ cu. ft. *Ans.*

7. Denote altitude of the entire pyramid by x. $\therefore x : x + 10 = 4 : 8$. $\therefore x = 10$ in. $\therefore$ vol. of frustum $= 373\frac{1}{3}$ cu. in. *Ans.*
Vol. of entire pyramid $= 426\frac{2}{3}$ cu. in. *Ans.*

8. Vol. of frustum $= 333$ cu. in. $\therefore 5.26+$ in. *Ans.*

9. Denote the edge of the second cube by x.
$\therefore x^2 : 10^2 = 5T : T$. $\therefore x = 10\sqrt{5}$ in. $= 22.36$ in. *Ans.*

10. 8 times. 4 times.

11. Denote the required volume by x.
$\therefore x : 400 = 21^3 : 12^3$. $\therefore x = 2143\frac{3}{4}$ cu. ft. *Ans.*

12. $4 \times \$250 = \$1000$. *Ans.*

13. 27 times. 9 times. The relative cost of the larger size of package $= \frac{1}{3}$.

14. The strength of Goliath is 9 times as great (§ 578). His weight is 27 times as great (§ 578). His activity increases with his strength, but diminishes with his weight. $\therefore$ it is $\frac{9}{27}$, or $\frac{1}{3}$, as great.

PAGE 397

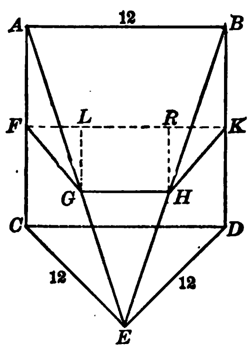
15. Denote the altitude of the frustum by x.
$\therefore 15 - x : 15 = 40 : 60$. $\therefore x = 5$ in. *Ans.*

16. $155\frac{1}{3}$ lb.

17. m is a rectangle 6×3 ft. $\therefore V = 220$ cu. ft. *Ans.*

18. m is a rectangle 65×29 ft. $\therefore V = 728.8+$ loads. *Ans.*

19.



Let AED be the given prismatoid, with FH the midsection. Then $FK = 12$ ft., $FG = GH = HK = 6$ ft. (§§ 300, 304). Draw GL, RH, FK. $\therefore LR = 6$ ft., $FL = 3$ ft., $GL = \sqrt{\overline{FG}^2 - \overline{FL}^2} = \sqrt{36 - 9}$ ft. $= 3\sqrt{3}$ ft. Area $FGHK = \frac{3\sqrt{3}}{2}(12 + 6)$ sq. ft. $= 27\sqrt{3}$ sq. ft. $\therefore V = \frac{15}{6}\left[0 + \frac{144\sqrt{3}}{4} + 4(27\sqrt{3})\right]$ cu. ft. $= 360\sqrt{3}$ cu. ft. $= 623.53+$ cu. ft. *Ans.*

20. m is a rectangle whose dimensions are $\frac{a}{2}$ and $\frac{b + c}{2}$. $\therefore V = \frac{ah}{6}(2b + c)$. *Ans.*

21. Might occur as a cut for a railroad track.

1. §§ 513, 515, 526, 528, 531, 550, 551, 552.
2. §§ 519, 527, 530.
3. §§ 522, 532, 548.
4. §§ 524, 538, 540, 541, 542, 543, 544, 546.
5. §§ 553, 555, 565, 566, 569, 570, 571.
6. §§ 557, 558, 560.
7. §§ 561, 562, 564, 573.
8. §§ 577, 578.
9. No. Yes.
10. A quadrangular pyramid. A frustum of a regular pyramid. A cube. A rectangular parallelopiped. A tetrahedron.
11. § 329. § 338. §§ 526, 560.

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12. § 534. See Ex. 1 in Group 102.

PAGE 403

1. Cylinder of revolution.
2. § 456, § 146.

PAGE 405

1. 440 sq. in. 748 sq. in.
2. $2\pi r^2$, $4\pi r^2$. $2\pi h^2$, $4\pi h^2$.
3. $4\pi r^2$, $6\pi r^2$. πh^2, $\frac{3}{2}\pi h^2$.
4. $2h : r$; $h : r + h$.
5. $46.93.
6. 4.00+ sq. ft.
7. 353.5+ sq. ft.

   $$\text{Saving} = \frac{3 \times 540 \times 300 \times \$3.75}{4 \times 2000} = \$227.81. \quad \textit{Ans.}$$

PAGE 406

2. 6285.7+ sq. ft.
3. 95.04 sq. in. 122.76 sq. in.
4. 33 sq. ft. $41\frac{5}{8}$ sq. ft.

PAGE 407

5. 93.86 sq. ft. 138.55+ sq. ft.
6. 492.8 sq. ft. 689.92 sq. ft.
7. 6600 sq. ft.
8. 1 sq. yd. = 1296 sq. in. Let x = radius of the base. $2\pi(12x) = 1296$. $\therefore x = 17.18$ in. *Ans.*
9. Denote the altitude by x. Then $2\pi\, 10\,(10 + x) = 900$. $\therefore x = 4.318+$ ft. *Ans.*

10. $r = \dfrac{S}{2\pi h}$.

11. $h = \dfrac{T}{2\pi r} - r$.

12. Find the value of r in $S = 2\pi rh$ $\left(\text{i.e., } r = \dfrac{S}{2\pi h}\right)$ and substitute for r in $T = 2\pi r\,(r + h)$.

$\therefore T = \dfrac{S^2 + 2\pi h^2 S}{2\pi h}$.

13. 3 times.

14. 275 sq. ft.

16. $\dfrac{h}{r} = \dfrac{h'}{r'}$ (§ 588). $\therefore \dfrac{h + r}{r} = \dfrac{h' + r'}{r'}$, or $\dfrac{h + r}{h' + r'} = \dfrac{r}{r'}$ (§§ 286, 284).

$$\frac{T}{T} = \frac{2\pi r(h + r)}{2\pi r'(h' + r')} = \frac{r}{r'} \times \frac{h + r}{h' + r'} = \frac{r}{r'} \times \frac{r}{r'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}.$$

17. $20 \times 5000 + \frac{22}{7}\,(10)\,5000 = 257{,}142.8 +$. $\therefore$ 257,142.8 + sq. ft. *Ans.*

18. §§ 180, 583, 585.

20. § 599.

21. The total surface of a cylinder of revolution equals the product of the circumference of its base by the sum of its base and altitude.

PAGE 408

1. § 602.

2. $21\frac{7}{18}$ cu. ft. $256\frac{2}{3}$ board ft.

3. 5.2.

PAGE 409

1. $246\frac{2}{3}$ cu. in.

2. 10,472 cu. ft.

3. $70\frac{2}{3}$ cu. ft.

4. 274.89 cu. in.

PAGE 410

5. $141\frac{3}{7}$ bbl.

6. 5892.9 – cu. yd.

7. Denote its height by x.

Then $\pi\left(\dfrac{9x}{4}\right) = \dfrac{231}{16}$. $\therefore x = 2.04+$ in. *Ans.*

8. 31.24 + mi.

9. $\pi\left(\dfrac{x^2}{4}\right) x = 231$. $\therefore x = 6.64+$ in. *Ans.*

10. Denote the volumes by V and V'.

$\therefore V : V' = (\pi r^2)r' : (\pi r'^2)\,r = r : r'$.

11. From $V = \pi r^2 h$, $r = \sqrt{\dfrac{V}{\pi h}}$. *Ans.*

Also $S = 2\pi rh$, $h = \left(\frac{S}{2\pi r}\right)$. Substitute for h in $V = \pi r^2 h$.

$\therefore V = \pi r^2 \left(\frac{S}{2\pi r}\right) = \frac{rS}{2}$. *Ans.*

12. 96.1+ cu. in.

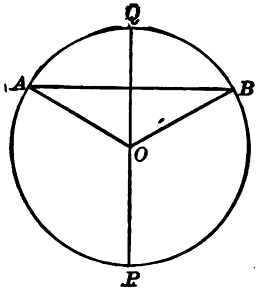
13. 1299+ gal.

14. $4022\frac{6}{7}$ lb.

15. 4 : 5; 8 : 125.

16. 26.78+ gal.

17.



Let circle O represent an end of the tank.

Area $\triangle OAB = \left(\frac{15\sqrt{3}}{4}\right)\frac{15}{4}$ sq. in. = 24.35+ sq. in.

Area sector $AOBP = \frac{2}{3}\left(\frac{22}{7}\right)\left(\frac{15}{2}\right)\left(\frac{15}{2}\right)$ sq. in. = 117.85+ sq. in.

Area of segment ABP = 142.20+ sq. in.

No. gallons $= \frac{35 \times 142.2+}{231} = 21.54+$. *Ans.*

18. 64.

PAGE 413

2. Convex surfaces of two cones and one cylinder.

3. 12.3+ in. **4.** 1200 lb. **5.** 68.06 in.

6. $\pi 9h = 512$ cu. in. $\therefore h = 18.1+$ in. *Ans.*

PAGE 414

1. Isosceles.

2. The plane cannot have two elements in common with the conical surface, for if it did, the line tangent to the base would become a secant.

PAGE 418

1. 154 sq. ft. 229.46 sq. ft.

2. 19.7274 sq. in. 26.5188 sq. in.

3. 33 sq. ft. $40\frac{1}{14}$ sq. ft.

4. $204\frac{2}{7}$ sq. in. $282\frac{6}{7}$ sq. in.

5. 54.54+ sq. yd. (§ 625).

6. Denote the altitude by x. Hence $400 = \pi\, 10 \sqrt{x^2 + 10^2}$.
$\therefore x = 7.89-$ yd. *Ans.*

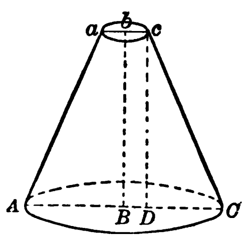
7. Denote the radius of the base by x. Then $l = \sqrt{x^2 + 100}$. The area of the base $= \pi x^2$. $\therefore \pi x \sqrt{x^2 + 100} = 11\pi x^2$.
$\therefore x = .912+$ ft. *Ans.*

8. From $S = \pi r l$, $r = \dfrac{S}{\pi l}$. Substitute for r in $T = \pi r\,(r + l)$.

$\therefore T = \dfrac{S^2 + \pi S l^2}{\pi l^2}$. *Ans.*

9. Solving $T = \pi r\,(r + l)$ with respect to r, $r = -\dfrac{l}{2} \pm \sqrt{\dfrac{T}{\pi} + \dfrac{l^2}{4}}$.
Ans.

10.



Let aC be the given funnel, with altitude bB, ac the diameter of the upper base, and AC that of the lower base. Draw $cD \perp AC$. Then $DC = BC - bc = 4 - 1 = 3$. Then $l = cC = \sqrt{\overline{cD}^2 + \overline{DC}^2}$ $= \sqrt{7^2 + 3^2}$ ft. $= \sqrt{58}$ ft. $\therefore$ by § 628, $S = \dfrac{\pi \sqrt{58}\,(4 + 1)}{144}$ sq. ft. $= .831+$ sq. ft. *Ans.*

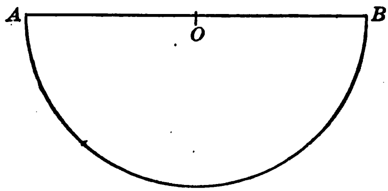
11. In the figure to Ex. 10, let $BC = r_1$, $bc = r_2$, $\angle\, cCA = 45°$. $\therefore DC = r_1 - r_2$. But $\angle\, DcC = 45°$ (§ 106). $\therefore DC = r_1 - r_2$. $\therefore cC = l = (r_1 - r_2)\sqrt{2}$. $\therefore S = \pi\,(r_1 - r_2)\sqrt{2}\,(r_1 + r_2) = (r_1^2 - r_2^2)\,\pi\sqrt{2}$. *Ans.*

PAGE 419

12. A sector of a circle.

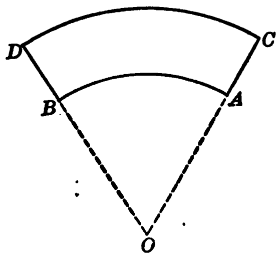
13. In an equilateral cone (whose slant height is l) the radius of the base is $\frac{1}{2}\, l$ (§ 633). $\therefore$ circf. of base $= 2\pi\,(\frac{1}{2}\, l) = \pi l$, or the semi-

circumference of a circle whose radius is l (§ 383). Hence cut out a semicircle of cardboard, whose radius = slant height of the



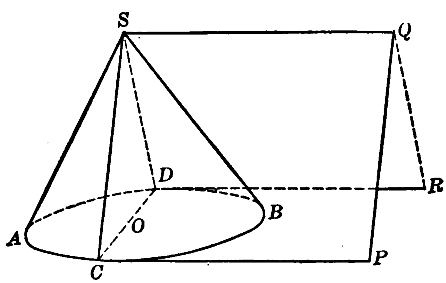
required cone, and roll up the semicircle till its bounding radii coincide (OA with OB).

14.



Use that part of a circular ring cut out by two radii. Let AB and CD be arcs of concentric circles included between the radii OAC and OBD. Roll up $ACDB$ till AC coincides with DB.

15.



The intersection is a straight line (§ 439, (2)) and this line passes through the vertex of the cone, for the vertex of the cone lies in both tangent planes (§ 687) and hence must lie in the line common to them.

PAGE 420

1. 62.85+, 113.13+, 50.28+.

2. $l = r\sqrt{2}$; $\pi r^2\sqrt{2}$, $\pi r^2(1+\sqrt{2})$, $\frac{\pi r^3}{3}$.

3. (1) radius of sector = slant height = 13 in.

   (2) central $\angle = \frac{2\pi 5}{2\pi 13}$ of $360° = 138° 27' 41''+$. *Ans.*

PAGE 421

1. 8.08+ cu. in.

2. 9900 cu. in.

3. $146\frac{2}{3}$ cu. ft.

4. $314\frac{2}{7}$ cu. in.

5. $r = \frac{44}{2\pi}$ ft. = 7 ft. $\therefore V = 246\frac{2}{3}$ bu. *Ans.*

6. 49+.

7. $h = \frac{l\sqrt{3}}{2}$, $r = \frac{l}{2}$. $\therefore \frac{\pi}{3}\left(\frac{l^2}{4}\right)\frac{l\sqrt{3}}{2} = \frac{231}{4}$ cu. in.

   $\therefore l = 3.169+$ in. *Ans.*

8. $h = \sqrt{l^2 - r^2}$. $\therefore V = \frac{\pi r^2\sqrt{l^2 - r^2}}{3}$. *Ans.*

   From $S = \pi rl$, $l = \frac{S}{\pi r}$. Substitute for l in the above formula for V. $\therefore V = \frac{1}{3}\pi r^2\sqrt{\frac{S^2}{\pi R^2} - r^2}$. *Ans.*

9. 2156 cu. ft.

10. Use figure of Ex. 10, p. 418. Draw $cD \perp BC$. Then $DC = BC - bc = 9$ ft. $- 6$ ft. $= 3$ ft. $\therefore cD = \sqrt{\overline{cC}^2 - \overline{DC}^2} = \sqrt{25 - 9}$ ft. $= 4$ ft. $\therefore V = 716.56+$ cu. ft. $\therefore$ cost = \$358.28. *Ans.*

11. By § 632, $88 = \frac{\pi 9}{3}(4r^2 + r^2 + 2r^2)$. $\therefore r = \frac{2}{3}\sqrt{3}$ ft. = 1.154+ ft. *Ans.*

12. $x^3 = \frac{\pi 6}{3}(81 + 16 + 36)$ cu. ft. $\therefore x = \sqrt[3]{836}$ ft. = 9.420 + ft. *Ans.*

13. Use the figure of Ex. 15, p. 419.

    Let S–ACD be the given circular cone and CD a diameter of the base. Let SR be a tangent plane through the element SD, and SP a tangent plane through SC. Let SP and SR intersect in the line SQ. Then $CP \parallel DR$ (§§ 210, 92). Construct the plane $PQR \perp CP$ (§ 442). $\therefore DR \perp$ plane PQR (§ 462). $\therefore$ planes SP and $SR \perp$

plane PQR (§ 482). $\therefore SQ \perp$ plane PQR (§ 485). $\therefore SQ \parallel CP$ (§ 463). $\therefore SQ \parallel$ plane ACB (§ 451).

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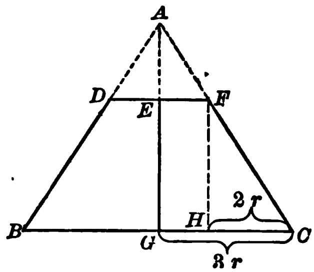
14.



Let ABC be a section through the axis of the equilateral cone and let $DFCB$ be a section of the frustum. $\therefore AC = 6r,\ AF = 2r.$ $\therefore FC = AC - AF = 4r.$ $\therefore FH = \sqrt{\overline{FC}^2 - \overline{HC}^2} = \sqrt{16r^2 - 4r^2}$ $= 2r\sqrt{3}.$ $\therefore$ Volume of frustum $= \dfrac{26\sqrt{3}\pi r^3}{3}$ (§ 632). *Ans.*

15. The upper segment of the altitude is to the entire altitude as $1 : \sqrt[3]{2}$ (§ 626).

16. The radii of the two bases and the altitude (or slant height).

17. Four times: eight times as much.

18. 9 : 27.

19. $a = \pi r^2.$ $\therefore r = \sqrt{\dfrac{a}{\pi}}.$ Also $\pi r l = b.$

$\therefore l = \dfrac{b}{\pi r} = \dfrac{b}{\pi}\left(\dfrac{\sqrt{\pi}}{\sqrt{a}}\right) = \dfrac{b}{\sqrt{a\pi}}.$ $\therefore h = \sqrt{\dfrac{b^2}{a\pi} - \dfrac{a^2}{a\pi}} = \sqrt{\dfrac{b^2 - a^2}{a\pi}}.$

$V = \frac{1}{3}\sqrt{\dfrac{ab^2 - a^3}{\pi}}.$ *Ans.*

20. 656.2+ sq. in. 628.57+ cu. in.

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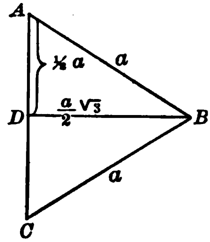
4. Area of surface described by $AB = \pi\left(\dfrac{a\sqrt{3}}{2}\right)a = \dfrac{\pi a^2\sqrt{3}}{2}$ (§ 625).

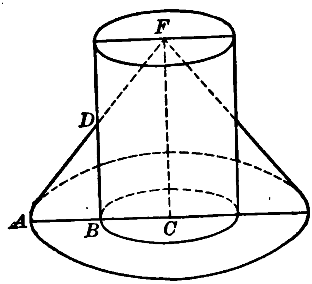


Area of surface described by $AB + BC = \pi a^2\sqrt{3}$. *Ans.*

5. Denote the surfaces by S and S' and volumes by V and V'. $\dot{S} : S = 2\pi\frac{a}{2}b : 2\pi\frac{b}{2}a = 1 : 1$. $\therefore S = S'$ (§ 600).

Also, $V : V' = \pi\frac{a^2}{4}b$. $\therefore \pi\frac{b^2}{4}a = a : b$ (§ 603).

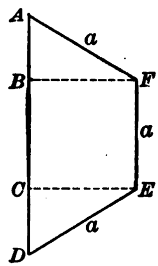
6.



Let BF be the given cylinder with axis FC, and FAC the cone, and D a point on the locus. Then the $\triangle$ ACF and ABD are similar rt. $\triangle$ (§ 305). $DB : FC = AB : AC$ (§ 302). But $AC = 2\,AB$ (Hyp.). $\therefore DB = \frac{1}{2}FC$. $\therefore$ all the points on the locus are at a distance $= \frac{1}{2}FC$ from the plane of the base. $\therefore$ they lie in the same plane (§ 465), $\therefore$ the locus is a circle (§594), whose plane bisects the altitude.

7. Let AD be the diagonal used as an axis, and AF, FE, ED, the sides of the hexagon. Then $\triangle$ ABF and DCE generate cones of revolution, and $BCEF$ a cylinder of revolution. $BF = \frac{a}{2}\sqrt{3}$, $AB = \frac{1}{2}a$.

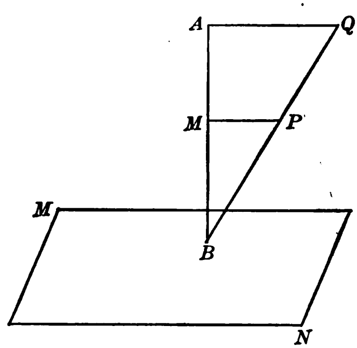
$$\left.\begin{aligned}\text{Surface } AF &= \pi\left(\frac{a}{2}\sqrt{3}\right)a.\\ \text{Surface } FE &= 2\pi\left(\frac{a}{2}\sqrt{3}\right)a.\end{aligned}\right\}\ \therefore \text{ surface } AFED = 2\pi\sqrt{3}a^2.\quad \textit{Ans.}$$



$$\left.\begin{aligned}\text{Vol. } ABF &= \frac{\pi a^2 3}{4}\times\frac{a}{6} = \frac{\pi a^3}{8}.\\ \text{Vol. } BFEC &= \frac{\pi a^2 3}{4}\cdot a = \frac{3\pi a^3}{4}.\end{aligned}\right\}\ \therefore \text{ volume } AFED = \pi a^3.\quad \textit{Ans.}$$

8. The surface of a cylinder of revolution, with the given line as axis and the given distance as radius (§§ 579, 586).

9.



Let MN be given plane, and AB a line $\perp MN$. Let the point P move so that its distance from MN is to its distance from AB as $m : n$. Let it move first in the plane ABQ and let P and Q be two of its positions in that plane, and let PM and $QA \perp AB$.
$MB : MP = m : n$. Also $BA : AQ = m : n$. $\therefore MB : MP = BA : AQ$. But $\angle BMP = \angle BAQ$ (being rt. $\angle$s). $\therefore \triangle BMP$

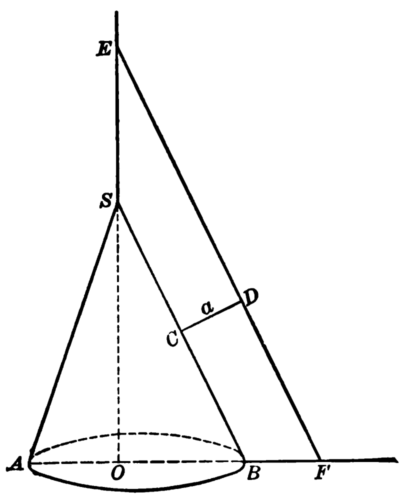
and BAQ are similar (§ 310). $\therefore < MBP = < ABQ$. $\therefore P$ lies in the line BQ. $\therefore$ the locus in the plane ABQ is the hypotenuse of a rt. $\triangle$ of which one leg is AB. Hence the locus in space is the surface of a cone of revolution whose axis is AB (§ 612).

10. The surface of a cone of revolution of which the given line is the axis (§ 612).

11. The surface of a cone of revolution whose axis is the $\perp$ to the plane at the given point. See Ex. 10.

12. Let R be the radius of the given cylinder, and a = given distance. Then the required locus is the surface of a cylinder of revolution whose radius is $R + r$, and whose axis is the axis of the given cylinder. (See Ex. 8).

13.



Let S–AB be the given cone, and CD the given distance $\perp CB$. Find first the locus of D in the plane SOB. This will be the line $EF \parallel SB$, meeting the axis produced in E, and the base produced in F. Hence the locus in space of the point D will be the surface of a cone of revolution generated by revolving the rt. $\triangle$ EOF about the leg OE as an axis (§ 612).

14. V.F.C. $= \dfrac{\pi a^2 h}{3}$; V. of cone $= \dfrac{\pi a^2 h}{3}$. $\therefore 7 : 6$. *Ans.*

15. The volume generated by one $\triangle$ is a cone. The volume generated by the other triangle = (vol. of cylinder generated by whole rectangle) − (vol. of cone) = 2 vol. cone. $\therefore \frac{1}{3}\pi b^2a : \frac{2}{3}\pi b^2a = 1 : 2$. *Ans.*

PAGE 424

1. §§ 581, 582, 590, 593.
2. §§ 583, 586, 591, 592, 599, 600, 601, 602, 603, 604.
3. §§ 608, 620.
4. §§ 612, 614, 621, 624, 625, 626, 629, 630, 631.
5. §§ 627, 628, 632.
8. Two.
9. Two.
10. Three.
11. Three.
12. Yes, the altitude and circumference of the base.

PAGE 428

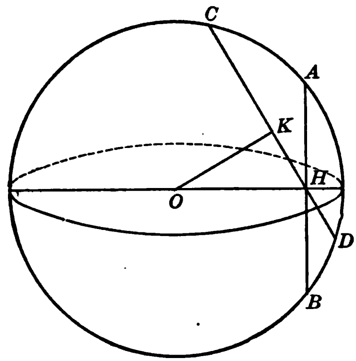
1. 5 in.
2. The equator and meridians of longitude. The parallels of latitude.
3. Two. Three points determine a plane.

PAGE 432

1. Earth, sun, moon, planets, their satellites, the stars.
2. The axis of each circle must be in the plane of the other circle (§ 486).

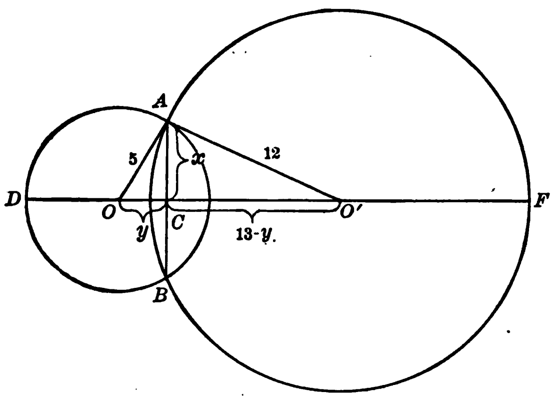
PAGE 435

1. The center of the sphere is equidistant from all points on the small circle and hence lies in the $\perp$ to the plane of the small circle at its center (§ 448). Hence this $\perp$ passes through the center of the sphere. Or use § 440, after passing two planes through the line joining the two centers.
2. 32 in.
3. 5 in.
4. Let O be a sphere in which AB is the diameter of a circle through the point H and $\perp OH$, and CD the diameter of any other circle through H. Draw $OK \perp CD$. Then in rt. $\triangle OKH$, $OH > OK$ (§ 135). $\therefore CD > AB$ (§ 209). $\therefore CK > AH$ (Ineq. Ax. 2).



Area of circle having diameter $CD >$ area of circle having diameter AB (§ 393).

5.

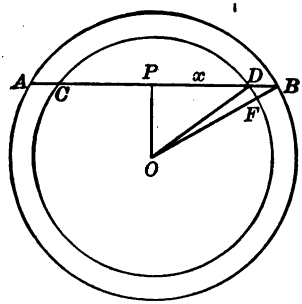


Let O and O' be the centers of the two spheres. Draw the common chord AB ($\perp OO'$ by § 221). In the rt. $\triangle OAC$, $x^2 = 25 - y^2$. In the rt. $\triangle O'AC$, $x^2 = 144 - (13 - y)^2$. $\therefore 25 - y^2 = 144 - (13 - y)^2$. $\therefore y = \frac{25}{13}$. $x^2 = 25 - \frac{625}{169} = \frac{3600}{169}$.
$\therefore \pi x^2 = 66.9485+$ sq. in. *Ans.*

6. Let AB be the section made by the cutting plane.

$$OP = 10 \text{ in.}, OB = 20 \text{ in.}$$
$$OD = OF = 18 \text{ in.}$$
$$PD = x, PB = y.$$

$$\therefore y^2 = \overline{20}^2 - \overline{10}^2 = 300.$$
$$x^2 = 18^2 - 10^2 = 224.$$



Area of ring $= \pi\,(y^2 - x^2) = 238.8+$. $\therefore$ 238.8+ sq. in. *Ans.*

7. Rapid and accurate location of places on the earth's surface; aids also in comparing locations, making maps, determining and comparing local time, etc.

8. All the planes mentioned are $\perp$ the axis of the great circle (§§ 646, 662) and $\therefore \parallel$ (§ 455).

PAGE 437

1. The locus is a circle. It becomes a point when the $\perp$ distance to the plane from the midpoint of the line $= \frac{1}{2}$ the given line. It disappears when this $\perp$ is greater than $\frac{1}{2}$ the given line.

2. Pass a plane through any point (outside the spheres) in the plane, and the centers of the two spheres. Use the method of demonstration employed for Ex. 9, p. 212.

3. The diagonal of the square $(= 12\sqrt{2})$ is the diameter of the small circle of the sphere in which the square is inscribed. $\therefore$ distance $=$ $\sqrt{\overline{20}^2 - (6\sqrt{2})^2} = \sqrt{328}$. $\therefore$ 18.11 + in. *Ans.*

4. A small circle of the sphere.

5. A sphere whose diameter is the line connecting the two points.

6. R and r are $\perp$ the common tangent of the two circles (§ 210). $\therefore$ ▲ on the figure are similar (§ 305). $\therefore l : l + d = r : R$, etc. $l =$ 865,268+ mi.

7. 232,042+ mi.

8. When the moon's distance from the earth equals or is less than the length of its shadow, the eclipse is total. When greater, the eclipse is annular.

PAGE 439

3. Use §§ 663, 442. That is, draw a radius of the sphere to the given point, and at the given point construct a plane ⊥ the radius.

4. The diagonal of the base of the pyramid is $3\sqrt{2}$. Denote the altitude of the pyramid by h.

   Then $h : \frac{3\sqrt{2}}{2} = \frac{3\sqrt{2}}{2} : 24 - h$ (§ 320).

   $\therefore h = 23.811+$, or $.189-$.

   $\therefore V = 71.433+$ cu. in., or $.567+$ cu. in. *Ans.*

5. The cones are similar.

   $\therefore h^3 : h'^3 = V : V' = 3 : 4$, or $h : h' = \sqrt[3]{3} : \sqrt[3]{4}$.

6. $2\pi r = 44$. $\therefore r = 7$. $\therefore V = 2156$ cu. in. *Ans.*

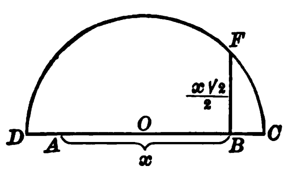
PAGE 440

1. Denote the radius of the section by x.

   $x^2 = \overline{20}^2 - \overline{5.6}^2 = 368.64$.

   Area of section $= \pi x^2 = 1158.58 +$ sq. in. *Ans.*

2.



   Denote a side of the inscribed cube by x. Then a diagonal of its face $= x\sqrt{2}$.

   $DB = r + \frac{1}{2}x$, $BC = r - \frac{1}{2}x$.

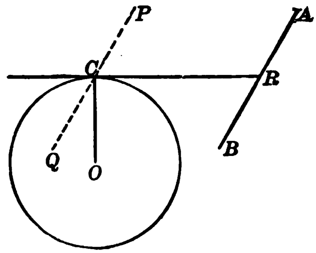
   $\therefore (r + \frac{1}{2}x)(r - \frac{1}{2}x) = \frac{2x^2}{4}$ (§ 320).

   $\therefore x = \frac{2r}{\sqrt{3}}$ and $x^3 = \frac{8r^3\sqrt{3}}{9}$. *Ans.*

3. See the solution of Ex. 5, p. 435. The locus is a circle whose radius is $13.90 +$ in., and whose plane is $5\frac{5}{8}$ in. from B.

4. Let O be the sphere and AB the given line. Draw a plane through $O \perp AB$ and intersecting AB in R (§ 443). From R draw a tangent RC, to the circle in which the plane intersects the sphere. Draw

a plane through AB and RC. This will be the plane required. For through C in the plane CAB draw $PQ \parallel AB$. $PQ \perp$ plane



OCR (§ 462). $\therefore PQ \perp OC$ (§ 436). $\therefore OC \perp$ plane CAB. $\therefore$ plane CAB is tangent to the sphere (§ 686).

PAGE 445

1. They intersect.
2. 2 : 1.

PAGE 446

1. § 79.
2. § 80. § 83.
3. § 84.

PAGE 447

1. Bisect three of the dihedral ∠s of the given tetrahedral ∠ by constructing and bisecting their plane ∠s. Use § 493.
2. Use method of § 86.
3. § 95.
4. Diameter of the sphere ⊥ these planes.

PAGE 454

1. 54 spherids.
2. 99.28+ sq. in.

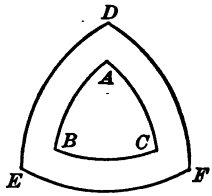
PAGE 456

1. 122.7°, 114.85°, 142.86°.
2. 152° 42′, 142° 37′, 117° 44′ 30″.
3. 122° 41′ 28″, 137° 46′ 37″, 142° 16′ 4″.
4. 90°.
5. The shortest route is the (minor) arc of the great circle through the two places (§ 689). Since the plane of the great circle passes through the center of the sphere, its arc cannot coincide with the arc of the small circle.

6. The largest possible size of the third size is just less than 150° (§ 688). The smallest possible size is just greater than 10°. For if $70° + x > 80°$, $x > 80° - 70°$ (Ineq. Ax. 1).

7. § 660.

8.



Let ABC be a spherical $\triangle$ in which $AB = AC$, and let DEF be the polar $\triangle$ of ABC. Then $\angle E = \angle F$ (§§ 680, 66). $\therefore DE = DF$.

9. Use the figure of Ex. 9. Let $\angle B = \angle C = 90°$; to show that $\angle E = \angle F = 90°$. Thus, $DE = DF = 90°$ (§ 680). $\therefore D$ is the pole of EF (§ 679). $\therefore \angle E = \angle F = 90°$ (§ 674).

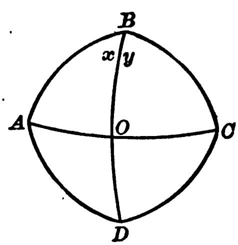
10. For the polar $\triangle$ is trirectangular by Ex. 9 and Ax. 1. Hence it equals the original $\triangle$ (§ 687).

11. Each triangle has two rt. $\angle$s (§ 693) which are equal. If the third $\angle$s are equal, the $\triangle$s are = (§ 687).

12. Use the figure p. 449. For the arc $AB' = 90°$ and arc $AC' = 90°$ (§ 678). $\therefore OA \perp$ plane $OB'C'$ (§ 660).

13. $\angle OAB = \angle O'BA$ (Hyp.) $\therefore \angle OBA = \angle O'AB$ (supplements of equal $\angle$s, § 66). $AB = AB$ (Ident.). $\therefore \triangle OAB = \triangle O'AB$ (§ 685).

14.



Let $ABCD$ be the given quadrilateral with the diagonals AC and BD intersecting at O.

$\triangle ABD = \triangle BCD$ (§ 686).

$\therefore \angle x = \angle y.$
$\therefore \triangle ABO = \triangle CBO$ (§ 684).
$\therefore \angle BOA = \angle BOC$, etc.

15. The equal sides must each be less than a semicircle. For by § 648, 4, if two great circles intersect they bisect each other.

16. $E = 99°$. $2 \angle A = 150°$. $\therefore \frac{\triangle}{\text{lune}} = \frac{99}{150} = \frac{33}{50}$. *Ans.*

PAGE 459

1. 1257.1+ sq. in. **2.** 314.28+ sq. in. **3.** 188.5+ sq. in.

PAGE 460

1. By § 711, $S = \frac{4\pi \overline{14}^2}{144}$ sq. ft. = 17.11+ sq. ft. *Ans.*

2. 38.5+ sq. in. *Ans.* **3.** 509.1 sq. ft.

4. $4\pi r^2 = 616$ sq. in. $\therefore r = 7$ in. *Ans.*

5. $\pi d^2 = 1296$ sq. in. $\therefore d = 20.3$ in. *Ans.*

6. $2\pi r = 1$ ft. $\therefore r = \frac{1}{2\pi}$ ft. $\therefore S = 4\pi r^2 = \frac{4\pi}{4\pi^2} = \frac{1}{\pi}$. $\therefore S =$ .3183+ sq. ft. *Ans.*

7. $2\pi r^2 = 900$ sq. ft. $\therefore r = 23.93+$ ft. *Ans.*

8. By § 715, $S = \frac{\pi r^2 A}{90°} = \frac{\pi \overline{14}^2 \times 36°}{90°}$ sq. in. $\therefore S = 246.4+$ sq. in. *Ans.*

9. 64.02+ sq. in.

10. S of lune : S of sphere = 24° : 360°. $\therefore S$ of lune : 4 sq. ft. = 24° : 360°. $\therefore S = \frac{4}{15}$ sq. ft. *Ans.*

11. By § 716, $S = \frac{\pi r^2 E}{180°} = \frac{\pi 7^2 \times 110°}{180°}$ sq. ft. = $94\frac{1}{9}$ sq. ft. *Ans.*

12. 122.2.

13. $E = 135°$. Then S of $\triangle$: S of sphere = E : 720. S of $\triangle$: 10 sq. ft. = 135 : 720. $\therefore S = 1\frac{7}{8}$ sq. ft. *Ans.*

14. The $\angle$s of the polar $\triangle$ are 80°, 70°, 60°. $\therefore E$ of the polar $\triangle$ = 30°. $\therefore S = 134.1-$. *Ans.*

15. $3900 = \frac{\pi r^2 130°}{180°}$ (§ 716). $\therefore r = 41.45-$. *Ans.*

16. Then $E = 240°$. $\therefore$ the sum of the $\angle$s of the $\triangle$ = 240° + 180° = 420°. $\therefore$ each $\angle$ = 140°. *Ans.*

17. 264 sq. in.

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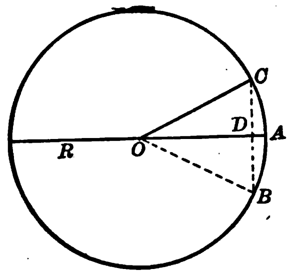
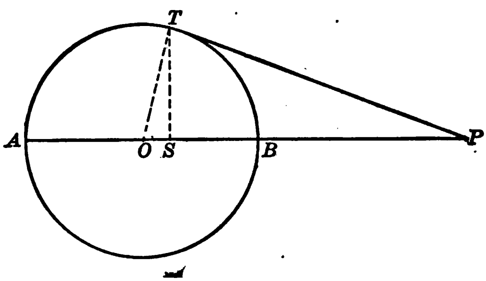
18. 45,257,142+ sq. mi.

19.



On the figure let OA be the plane of the earth's equator and C the position of Cairo. $\therefore \angle AOC = 30°$. Draw chord $CB \perp OA$, and the radius OB. Hence it follows that $\angle COB° = 60°$ and $\triangle OCB$ is equilateral, $CD = \frac{1}{2} CB = \frac{1}{2} r$. $\therefore S$ of zone between A and C $= 2\pi r\,(\frac{1}{2}r) = \pi r^2$. But the surface of a hemisphere $= 2\pi r^2$, etc.

20.



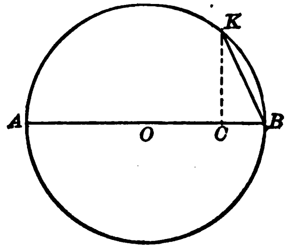
Let the sphere O represent the earth, and P the required point. Draw $TS \perp PO$, and the radius OT. Let the radius of the earth be denoted by r, and BP be x. Then $SB = \frac{2r}{3}$ (§ 714). $\therefore OS = \frac{1}{3}r$.
In the rt. $\angle\, OTP$, $OS : OT = OT : OP$ (§ 319),
or, $\frac{1}{3}r : r = r : r + x$. $\therefore x = 2r$. *Ans.*

21. $\frac{1}{6}$. *Ans.* See Ex. 20.

22. $2\pi rh = \pi r^2$. $\therefore h = \frac{r}{2}$. *Ans.*

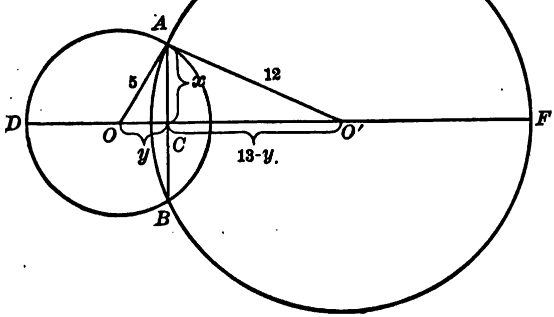
23. Let the sphere O represent the earth, and KB the chord of 3000 miles. Draw the diameter BA and $KC \perp AB$. The diameter of

the earth is taken as 8000 miles. $\therefore CB : KB = KB : AB$ (§ 320). Or, $CB : 3000 = 3000 : 8000$. $\therefore B = \frac{9000}{8}$ mi., or, $CB = \frac{9}{64}$ of



the diameter of the earth. S of zone $KB = \frac{9}{64}$ of earth's surface (§ 714).

24.



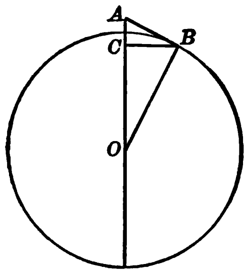
Let O and O' be the centers of the two spheres. Draw the common chord AB ($\perp OO'$ by § 221). In the rt. $\triangle OAC$, $x^2 = 25 - y^2$. In the rt. $\triangle O'AC$, $x^2 = 144 - (13 - y)^2$. $\therefore 25 - y^2 = 144 - (13 - y)^2$. $\therefore y = 1.92307+$ in. $\therefore DC = 6.92307+$ in. $\therefore S$ of zone $ADB = 2\pi 5 \times 6.92307+$ sq. in. $= 217.58+$ sq. in. *Ans.* $CF = 23.0779$ in. $\therefore S$ of zone $AFB = 1740.73+$ sq. in. *Ans.*

25. Denote the radius of the cylinder by x.

$\therefore r + x : x = x : r - x$. $\therefore x = \dfrac{r\sqrt{2}}{2}$.

S of cylinder $= 2\pi r^2$, or lateral area of the cylinder $= \frac{1}{2}$ area of the sphere.

26.



Let A be the position of the aviator, and AB be tangent a great circle of the earth. $\therefore AO = 4002$ mi. Let $BC \perp AO$.
$\therefore OC : OB = OB : OA$. $\therefore OC = 3998+$ mi., or the altitude of the zone visible to the aviator is $2 -$ mi.
$\therefore$ aviator sees $\frac{1}{4000}$ of earth's surface (approx.) $= 50{,}285+$ sq. mi. *Ans.*

27. 100 times as great. $\frac{1}{100}$ intensity. Yes. Because the intensity of every force radiated from a center varies inversely as the square of the distance from the center.

28. Since $36° = \frac{1}{10}$ of $360°$, the area of the $\triangle$ would be $\frac{1}{10}$ of a hemisphere, or $\frac{1}{20}$ of the whole sphere.
$= 10{,}057{,}142+$ sq. mi. *Ans.*

PAGE 465

1. $11{,}498\frac{2}{3}$ cu. ft.

2. $11{,}498\frac{2}{3}$ cu. in.

3. $22\frac{11}{24}$ cu. in.

4. $42.819+$ cu. yd.

5. .000,001,768 cu. ft.

6. 38,808 cu. in.

7. 260,224,450,000 cu. mi.

8. $\frac{\pi d^3}{6} = 1$. $\therefore d = 1.24+$ ft. *Ans.*

9. $r = 7$ in. $\therefore V = 1437\frac{1}{3}$ cu. in. *Ans.*

10. Denote the radius of the required sphere by x. Then $\frac{4}{3}\pi x^3 = \frac{4}{3}\pi 2^3 + \frac{4}{3}\pi 4^3$. Divide by $\frac{4}{3}\pi$. Then $x = \sqrt[3]{72}$ in. $= 4.160+$ in. *Ans.*

11. $\frac{4}{3}\pi r^3 = 4\pi r^2$. $\therefore r = 3$. *Ans.*

12. Diagonal of cube $= 6\sqrt{3}$. $\therefore r = 3\sqrt{3}$. $\therefore V = 587.9+$. *Ans.*

13. $V = \frac{\pi \overline{21}^3}{6} - \frac{\pi \overline{14}^3}{6}$ cu. in. $= 3413\frac{2}{3}$ cu. in. *Ans.*

14. Denote the radius of the outer surface by r, and that of the inner surface by r' Then $4\pi r^2 = 20\pi$. $\therefore r = \sqrt{5}$. $4\pi r'^2 = 12\pi$. $\therefore r' = \sqrt{3}$. $\therefore V$ of shell $= \dfrac{4\pi 5\sqrt{5}}{3} - \dfrac{4\pi 3\sqrt{3}}{3} = 25.07+$. *Ans.*

15. As a corollary from §§ 718, 705, it follows that the volume of a spherical wedge is to the volume of the sphere as the angle of the lune bounding the wedge is to 360°. Hence V of wedge : $\dfrac{4\pi 1000}{3}$ cu. in. $= 24° : 360°$. $\therefore V$ of wedge $= 279.3+$ cu. in. *Ans.*

16. By § 727, $V = \frac{2}{3}\pi\, 10^2 \times 2$ cu. in. $= 419+$ cu. in. *Ans.*

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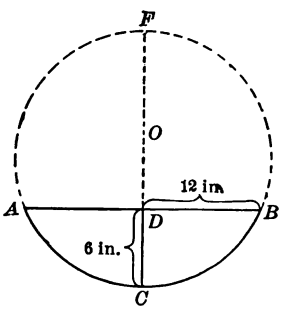
17. See formula 11, p. 493. $576.19+$ cu. in. *Ans.*

18.

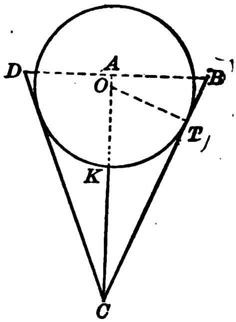


Let ABC be the given wash basin, and O the completed sphere of which it is a segment. Then $DF : DB = DB : DC$ (§ 320). Or, $DF : 12$ in. $= 12$ in. $: 6$ in. $\therefore DF = 24$ in. $\therefore OC$ or $r = 15$ in. By formula 12, p. 493, $V = \dfrac{\pi 36(15 - 2)4}{231}$ qt. $= 25.4-$ qt. *Ans.*

19. 5 : 11.

20. By formula 12, p. 493, $\pi 16\,(r - \frac{4}{3}) = 200$ cu. in. $\therefore r = 5.3105+$ in. *Ans.*

21. Let O be the given sphere and DBC the given wineglass. It is required to find the volume of the spherical segment with alt. AK. BC and DC are tangents to the circle O. $OT \perp BC$ (§ 210). $\therefore$ rt. $\triangle$ OTC and ABC are similar (§ 305). But $BC = \sqrt{\overline{AC}^2 + \overline{AB}^2}$ $= \sqrt{6^2 + \overline{2.5}^2}$ in. $= 6.5$ in. $BC : OC = AB : OT$, or, 6.5 in. $: OC$ $= 2.5 : 2$. $\therefore OC = 5.2$ in. $\therefore OA = 6$ in. $- 5.2$ in. $= .8$ in.



$\therefore AK = 2.8$ in. $\therefore$ vol. of spherical segment $ATK =$ $\pi(2.8)^2\left(2 - \frac{2.8}{3}\right)$ cu. in. $= 26.28+$ cu. in. *Ans.*

22. Let x and $3x$ denote the radii of the spheres.
$\therefore 4\pi x^2 + 36\pi x^2 = 40\pi$, etc.
$4\frac{4}{21}$ cu. in. $113\frac{1}{7}$ cu. in. *Ans.*

23. Denote a side of the $\triangle$ by a and the radius of the circle by r.

$\therefore r = \dfrac{a\sqrt{3}}{3}.$

T of cone $= \pi\left(\dfrac{a}{2}\right)\dfrac{3a}{2} = \dfrac{3\pi a^2}{4};$

S of sphere $= \dfrac{4\pi a^2}{3}.$ $\therefore 9 : 16.$ *Ans.*

V of cone $= \dfrac{\pi a^2}{12}\left(\dfrac{a\sqrt{3}}{2}\right) = \dfrac{\pi a^3\sqrt{3}}{4};$

V of sphere $= \dfrac{4\pi}{3}\left(\dfrac{a\sqrt{3}}{3}\right)^3 = \dfrac{\pi a^3\sqrt{3}}{27}.$

$\therefore 9 : 32.$ *Ans.*

24. $268{,}083.2+$ cu. yd. **25.** $\frac{1}{2}\sqrt[3]{6a^2b}$. **26.** 8192.

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1. Each volume = 216 cu. in. The surfaces are 216 sq. in. and 252 sq. in. Illustrates § 728 (2).

2. No. sq. yd. $= \dfrac{1{,}000{,}000\,(252 - 216)}{144 \times 9} = 2778-.$
Saving = \$555.56.

3. Each surface = 216 sq. in.
The volumes are 216 cu. in. and 201.6 cu. in. Illustrates § 728 (1).

4. T of cube $= 600$ sq. in.
$\therefore r$ of sphere $= \sqrt{\frac{600}{4\pi}} = \sqrt{\frac{150}{\pi}} = 6.908+$ in.
V of cube $= 1000$ cu. in.
V of sphere $= \frac{4\pi}{3}\left(\frac{150}{\pi}\right)(6.908+) = 1381.7+$ cu. in.
Illustrates § 729 (1).

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5. Each area $= 4\pi a^2$.
Volumes are πa^3 and $\frac{4}{3}\pi a^3$.
Illustrates § 729 (1).

6. T of cone $= 788.95$ sq. in.
S of sphere $= 616$ sq. in.
V of each $= 1437\frac{1}{3}$ cu. in.
Illustrates § 729 (2).

7. A cube. 8. A cube. 9. § 729 (2).

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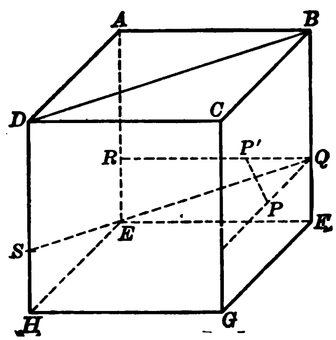
1.



Let AG be a cube and DB a diagonal of its upper base. Show $\angle ABD = \angle CBD$.
$\therefore$ dihedral $\angle A\text{–}BF\text{–}D =$ dih. $\angle D\text{–}BF\text{–}G$.
Let P be any point in the face $CBFG$. Through P draw $PP' \perp$ plane $DBFH$ and meeting face AF in P'. Through PP' pass a plane $\perp BF$ and intersecting BF in Q, and plane DF in SQ, etc.

2. Nine planes. One center.
3. Three planes. One center.
4. One or no plane. One center.
5. Four planes. No center.
6. Nine planes. One center.
7. Four planes. No center.
8. Infinite number of planes. No center.
9. Infinite number of planes. No center.
10. Infinite number of planes. One center.
11. To a plane. To a plane.
1. 3,298,909+.
2. Areas are equal. Volumes as 2 : 3.
3. $r = \sqrt{\frac{3a^3}{4\pi}}$.

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4. $\frac{4}{3}\pi r^3 = \frac{\pi 9 \times 12}{3}$. $\therefore r = 3$ in. *Ans.*
5. $3\,(5x) = \frac{4\pi 7^3}{3}$ cu. in. $\therefore x = 95.82+$ in. *Ans.*
6. Use the figure of Ex. 18, p. 466. $FC = 14$, $DC = 2$. $\therefore FD = 12$. $\therefore AC = \sqrt{24}$ (§ 320). $\therefore r$ of the cylinder $= 2\sqrt{6}$. Denote the altitude of the cylinder by x. Then $2\pi 2\sqrt{6}x = 2\pi 7 \times 2$. $\therefore x = \frac{7}{6}\sqrt{6} = 2.857+$. *Ans.*
7. $\therefore r$ is increased twice and V 8 times (§§ 713, 721).
8. 1012.5 yd. 37.5 yd.
9. 1 : 8. 1 : 64. 1 : 37 (approx.) 1 : 101 (approx.).
10. The volume of Jupiter is $\overline{11}^3$, or 1331, times as great as that of the earth (§ 721) and its surface $\overline{11}^2$, or 121, times as great (§ 713). The time required to radiate its heat will increase with the volume and diminish with the surface, or be $\frac{1331}{121}$, or 11, times as great.
11. $\sqrt{3} : 2$. 3 : 4.
12. Each $= 20\pi a^2$.
13. r of cylinder $= \frac{8}{3}$ in.
    $T = 145.2+$ sq. in. *Ans.*
14. 14.14+ in.

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15. The bridge will be 1,000,000,000 as heavy as the model and only 1,000,000 times as strong. Hence the power of the bridge to sustain itself will be only $\frac{1}{1000}$ that of the model.

1. §§ 644, 648 (3), 648 (4), 648 (5), 674.
2. §§ 645, 648 (2), 648 (6).
3. §§ 647, 648 (5), 648 (6), 650, 654, 655, 660.
4. § 669.
5. §§ 675, 677, 684, 685, 686, 687, 688, 691, 708, 709, 716.
6. §§ 673, 679, 680, 681.
7. §§ 662, 663, 664.
8. Two. Three. Three. Four.
9. Radius or diameter of the sphere. The circumference of a great circle.
10. Yes.
11. §§ 662, 210. §§ 654, 214.

PAGE 473

1. 1008 sq. in.; 1680 sq. in.; 4032 cu. in.
2. 1658.6+ sq. in.; 2274.6+ sq. in.; 7186.67− cu. in.
3. 341.5+ sq. in.; 521.5+ sq. in.; 756 cu. in.
4. 120 sq. in.; 147.712+ sq. in.; 89.88− cu. in.
5. 12 sq. ft.; 18 sq. ft.; $3\sqrt{3}$ cu. ft.
6. $694\frac{4}{7}$ sq. in.; 1188 sq. in.; $2803\frac{3}{7}$ cu. in.
7. r of the sphere $= 3$.
   $\therefore V$ of sphere $= 113\frac{1}{7}$. S of $\triangle = 22$. *Ans.*

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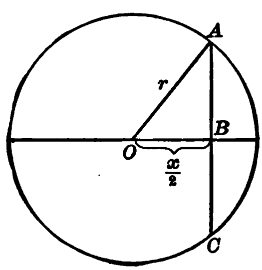
8. Denote the dimensions by $2x$, $3x$, $4x$. Then $4x^2 + 9x^2 + 16x^2 = (2\sqrt{29})^2$. $\therefore x = 2$. $\therefore T = 208$, $V = 192$. *Ans.*
9. $\frac{4}{3}\pi r^3 : 8r^3 = \pi : 6$.
10.



Denote a side of the cube by x. If AC is a diagonal of a face of the inscribed cube, $AB = \frac{x\sqrt{2}}{2}$.

$\therefore \left(\frac{x\sqrt{2}}{2}\right)^2 + \left(\frac{x}{2}\right)^2 = r^2. \quad \therefore x^2 = \frac{4r^2}{3}.$

$\therefore V$ of cube $= \frac{8r^3}{3\sqrt{3}}. \quad \therefore$ ratio $= \frac{4\pi r^3}{3} : \frac{3\sqrt{3}}{8r^3} = \pi : \frac{2}{3}\sqrt{3}.$

11. Each corner cut off is a pyramid with an isosceles right $\triangle$ (leg $= 2$) for base and 2 for its altitude, and $\therefore$ with $V = \frac{4}{3}$. $\therefore$ volume of remnant of cube $= 64 - 8\left(\frac{4}{3}\right) = 53\frac{1}{3}$. *Ans.*

12. Increases 2 times; 4 times; 8 times.

13. By § 633, $l = 2r$, $h = r\sqrt{3}$. $\therefore S = 2\pi r^2$, $V = \frac{\pi r^3\sqrt{3}}{3}$. *Ans.*

14. No. of bullets $= \dfrac{20 \times 8 \times 2}{\frac{\pi}{6}\left(\frac{3}{4}\right)^3} = 1448$. *Ans.*

15. Find the number of cubic feet in the frustum as if solid, the number of cu. ft. in the frustum which is the hollow interior; subtract, and multiply by 12. Thus no. bricks $= \frac{\pi 90}{3}\left(4^2 + \left(\frac{3}{2}\right)^2 + 6 - 2^2 - 1^2 - 2\right) 12 = 19{,}518$. *Ans.*

16. $300 = 2\pi r 6$. $\therefore r = 7.95+$. *Ans.*

17. $h = 2r$. $\therefore S = 4\pi r^2$; $T = 6\pi r^2$; $V = 2\pi r^3$. *Ans.*

18. $T = b^2 + 4\left(\frac{b^2\sqrt{3}}{4}\right)$. $\therefore b = \sqrt{\frac{T}{1+\sqrt{3}}}$. *Ans.*

19. Area of the base $= a^2$. $\therefore$ area of cross section $= \frac{a^2}{4}$ (§ 565, III). V of entire pyramid $= \frac{a^3}{3}$. V of part above the cross section is $\frac{1}{8}$ of this or $\frac{a^3}{24}$ (§ 578). $\therefore V$ of frustum below $= \frac{a^3}{3} - \frac{a^3}{24} = \frac{7a^3}{24}$. *Ans.*

20. $\frac{\pi}{6}(\overline{8050}^3 - \overline{8000}^3)$ cu. mi., or 10,183,000,000 cu. mi. *Ans.*

21. 21 : 32. *Ans.*

22. r of cylinder $= a$. $\therefore V = \pi(a^2)6a = 6\pi a^3$. *Ans.* V of sphere $= \frac{4}{3}\pi a^3$. *Ans.* V of pyramid $= \frac{4a^2 \times 6a}{3} = 8a^3$. *Ans.* r of cone $= a$. $\therefore V$ of cone $= 2\pi a^3$. *Ans.*

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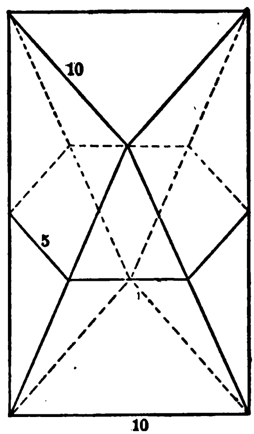
23. Use the figure of Ex. 20, p. 461. Denote BP by a, and OT by r.

$\therefore OP = a + r.$ $\therefore OS : r = r : r + a.$ $\therefore OS = \dfrac{r^2}{r+a}.$ $\therefore BS = r - \dfrac{r^2}{r+a} = \dfrac{ar}{r+a}.$ $\therefore$ surface of zone $TB = \dfrac{2\pi r\ (ar)}{a+r} = \dfrac{2\pi r^2 a}{a+r}.$ *Ans.*

24. $\dfrac{\pi r^2 E}{180°} = 12.$ $\therefore \dfrac{\pi 36E}{180°} = 12.$ $\therefore E = 19\tfrac{1}{11}.$ $\therefore$ sum of $\angle$s of $\triangle = 180° + 19\tfrac{1}{11}° = 199\tfrac{1}{11}°.$ $\therefore$ each $\angle = 66\tfrac{4}{11}°.$ *Ans.*

25.



$b_1 = b_2 = \dfrac{100\sqrt{3}}{4} = 25\sqrt{3}.$ Each side of midsection $= 5$ (§§ 300, 304). $\therefore m = 6\left(\dfrac{25\sqrt{3}}{4}\right) = \dfrac{75\sqrt{3}}{2}.$ $\therefore V = \tfrac{24}{6}(25\sqrt{3} + 25\sqrt{3} + 150\sqrt{3}) = 800\sqrt{3} = 1385.64+.$ *Ans.*

26. The driving power increases as the cubes of the dimensions, while the resistance increases only as their square. The carrying capacity increases as the volume, i.e., as the cube of the dimensions, whereas the resistance of the water ($\therefore$ cost of fuel) increases only as their square. Also the size of the crew, amount of wages, etc., are much less proportionately for the large vessel.

27. Use the figure of Ex. 9, p. 384. $\angle SBO = 60°.$ $\triangle SBD$ would be equilateral if line OD were drawn. But $OF = BF = 5.$ $\therefore OB = 5\sqrt{2}.$ $\therefore SB = 10\sqrt{2}.$ $\therefore SO = 5\sqrt{6}.$ $\therefore V = 408.25+.$ *Ans.*

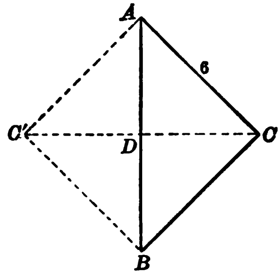
28. $V = \pi 8^2 \times 6$ cu. in. $= 1207 -$ cu. in. *Ans.*

29. See Ex. 10, p. 470.

30. Use the figure of Ex. 20, p. 461. $OT = 5$, $OS = 3$. $OP : OT = OT : OS$, or $OP : 5 = 5 : 3$. $\therefore OP = 8\frac{1}{3}$. $\therefore SP = 5\frac{1}{3}$. $\therefore PT = \sqrt{(\frac{25}{3})^2 - 5^2} = \frac{20}{3}$. $TS = \sqrt{25 - 9} = 4$. In the cone STP lateral area $= 83.33+$, and $V = 89.47+$. *Ans.*

31. $\frac{4}{3}\pi r^3 = \frac{4312}{3}$ cu. in. $\therefore r = 7$ in. $\therefore S = 616$ sq. in. *Ans.*

32.



Let $ACBC'$ be the given square and AB the diagonal about which it is revolved. Then ADC is an isosceles right $\triangle$. $\therefore AD = DC = 3\sqrt{2}$. AC and BC each generate the surface of a cone. $\therefore S = 36\pi\sqrt{2}$, $V = 36\pi\sqrt{2}$. *Ans.*

33. $8x^2\,(62.28) = 20{,}000$. $\therefore x = 6.33+$ ft. *Ans.*

34. No. of buckets $= \dfrac{18 \times \dfrac{9\sqrt{3}}{4}}{\pi(\frac{1}{4})\frac{3}{2}} = 59.5-$. *Ans.*

35. $2\pi rh = 440$ . . . . (1), $\pi r^2h = 1540$ cu. in. . . . . (2)
Divide (2) by (1), and find $r = 7$. Hence $h = 10$. *Ans.*

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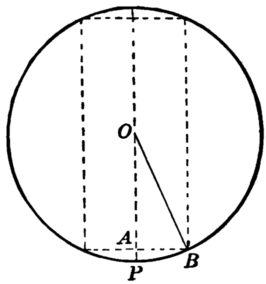
36. E of the spherical quadrilateral $= 60°$. $\therefore$ sum of $\measuredangle$ of the $\triangle = 180° + 60° = 240°$. $\therefore$ each $\angle$ of $\triangle = 80°$. *Ans.*

37. Denote the radius of cone by r. Then l of cone $= 2r$ and $h = r\sqrt{3}$ (see Ex. 13). $\therefore S = 2\pi r^2$, $V = \dfrac{\pi r^3\sqrt{3}}{3}$. Denote radius of cylinder by r'. $h = 2r'$. S of cylinder $= 2\pi r'\,(2r') = 4\pi r'^2 = 2\pi r^2$. $\therefore r'^2 = \dfrac{r^2}{2}$, $r' = \dfrac{r}{\sqrt{2}}$. $\therefore V$ of cylinder $= \dfrac{\pi r^2}{2} \cdot \dfrac{2r}{\sqrt{2}} = \dfrac{\pi r^3}{\sqrt{2}}$. V of cone : V of cylinder $= \dfrac{\pi r^3\sqrt{3}}{3} : \dfrac{\pi r^3}{\sqrt{2}} = \sqrt{6} : 3$. *Ans.*

38. No. of quarts $= \dfrac{\pi(\frac{3}{8})^2 \times 13 \times 12 \times 4}{231} = 1.194-$. *Ans.* No. of seconds $= (1.194-) \times 5 = 5.96+$. *Ans.*

39.



Let O be the given sphere and AB the radius of the auger hole. Then the part of the sphere bored out will be a cylinder of revolution capped by a segment of a sphere at either end. Half the altitude of the cylinder, OA, $= \sqrt{\overline{OB}^2 - \overline{AB}^2} = \sqrt{4^2 - (\frac{3}{2})^2}$ in. $= 3.708+$ in. $\therefore AP$, the altitude of one of the spherical segments $= 4$ in. $- 3.708+$ in. $= .292-$ in. $\therefore$ vol. bored out $= \pi\ (\frac{3}{2})^2(7.416+)$ cu. in. $+ 2\pi(.2928-)^2\left(4 - \dfrac{.292-}{3}\right)$ cu. in. $=213.66+$ cu. in. *Ans.*

40. h (of cone and cylinder) $= r$. $\therefore \dfrac{\pi r^2 . r}{3} : \dfrac{2\pi r^3}{3} : \pi r^2 \times r = 1 : 2 : 3$. *Ans.*

41. $r^3 : r'^3 = 8 : 125$ (§ 604). $\therefore r : r' = 2 : 5$ (§ 291). *Ans.*
10 in. $: r' = 2 : 5$. $\therefore r' = 25$ in. *Ans.*

42. Hence diameter of inner surface $= 28 - 4$ in. $= 24$ in.
V of shell $= \dfrac{\pi}{6}\ (\overline{28}^3 - \overline{24}^3)$ cu. in. $= 4257.5+$ cu. in. *Ans.*

1. The edges of the base reduce to 6 m., 7 m., 9 m., and h to 9 m. $S = 198$ sq. m., $T = 239.95+$ sq. m., $V = 188.78+$ cu. m. *Ans.*

2. The altitude is 17 dm. $S = 1360.3$ sq. dm., $T = 2260.3-$ sq. dm., $V = 5100$ cu. dm. *Ans.*

3. .02 m. = 2 cm. $\therefore S = 50.28+$ sq. cm., $V = 33.5+$ cu. cm. *Ans.*

4. 50 cm. = 5 dm. $S = 251.43-$ sq. dm., $T = 678.86$ sq. dm., $V = 616$ cu. dm. *Ans.*

5. $T = 288$ sq. cm., $V = 192\sqrt{3}$ cu. cm. *Ans.*

6. $S = 50.28$ sq. dm., $T = 75.42$ sq. dm., $V = 50.28$ cu. dm. *Ans.*

7. .02 m. = 2 cm. $\therefore$ area = 12.57 sq. cm. *Ans.*

8. 50 dm. = 5 m. $2\pi r$ = 5 m. $\therefore r = \frac{5}{2\pi}$ m. $\therefore S = 4\pi\left(\frac{5}{2\pi}\right)^2$ sq. m. = 7.954+ sq. m. *Ans.*

9. .25 m. = 2.5 dm. $\therefore V$ = 196.43+ cu. dm. = 196.43+ l. = (19.643+) (1.057) qt. = 207.626+ qt. *Ans.*

10. $r = \sqrt{\frac{175}{22}}$ dm. = 2.8203+ dm.
V = 94.01+ cu. dm. *Ans.*

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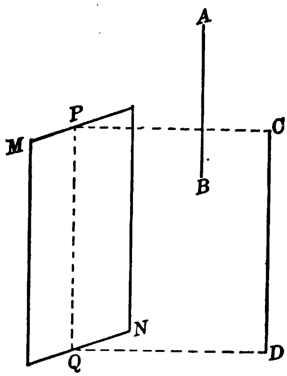
1.



Let the line AB and the plane MN each be $\parallel$ line CD. To prove $AB \parallel MN$. Through CD pass a plane intersecting MN in the line PQ. Then $CD \parallel PQ$ (§ 452). $\therefore AB \parallel PQ$ (§ 464). $\therefore AB \parallel$ plane MN (§ 451).

2. An edge of cube = $5\sqrt{2}$ in. $\therefore V = 250\sqrt{2}$ cu. in. = 353.55+ cu. in. *Ans.*

3. Using the given poles, describe an arc at a quadrant's distance from each pole (§ 678).

4. $AE \parallel BF$ (§ 463). $\therefore$ pass a plane through them (§ 434). Then plane $AEFB \perp$ plane MR (§ 482); also $\perp$ plane MN (§ 482). $\therefore MP \perp$ plane $AEBF$ (§ 485). $\therefore MP \perp EF$ (§ 436).

5. Through the other diagonal pass a plane $\perp$ given plane. Use §§ 487, 110.

6. $V = 226.28+$ cu. in. $T = 198$ sq. in. *Ans.*

7. 194.74+ lb. 8. $\frac{1}{3}$. 9. $\frac{3\pi l^2}{4}$.

10. The mean proportional $= \sqrt{h \times 2r}$. $\therefore$ area of circle $= \pi h \times 2r = 2\pi r h =$ lateral area of cylinder of revolution.

11. Denote the volumes by V and V', and the surfaces by S and S'. Then $V = \frac{1}{3} S \times r$, $V' = \frac{1}{3} S' \times r$. $\therefore V : V' = \frac{1}{3} S \times r : \frac{1}{3} S' \times r = S : S'$.

12. See Ex. 16, p. 385. 250 cm. $\doteq$ 25 dm. $S = 672$ sq. dm. $T = 868$ sq. dm. *Ans.*

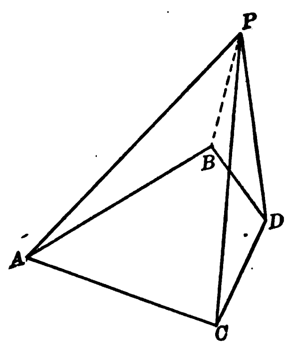
13. Use the figure p. 330. Let FC and HD be the given $\parallel$ lines, and FB and HB the planes through them, intersecting in AB. Construct a plane $PQ \perp HD$ (§ 442). Then $FC \perp PQ$ (§ 462). Planes FB and $HB \perp PQ$ (§ 482). $\therefore AB \perp PQ$ (§ 485). $\therefore AB \parallel FC$ and HD (§ 463).

14. $PH \perp EH$ and HF (§ 436). $\therefore$ rt. $\triangle PEH =$ rt. $\triangle PFH$ (§ 106) $\therefore PE = PF$. $\therefore \angle PEF = \angle PFE$ (§ 82).

15. $\frac{1}{48}$.

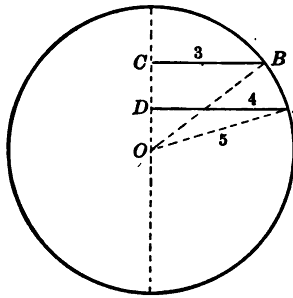
16. Through one of the given points draw a $\perp$ to the given plane (§ 443). Pass a plane through this $\perp$ and the other given point (§ 434). In the line of intersection of the two planes find a point such that the sum of its distances from the two given points is a minimum (Ex. 20, p. 174).

17.



$AB \perp$ plane PBD (Hyp.). Plane $ABCD \perp PBD$ (§ 482). $AC \perp$ plane PCD (Hyp.). $\therefore$ plane $ABCD \perp PCD$. $\therefore PD \perp$ plane $ABDC$ (§ 488).

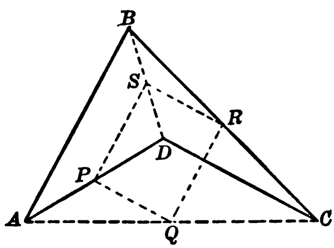
18.



$OC = \sqrt{\overline{OB}^2 - \overline{CB}^2} = \sqrt{25 - 9} = 4.$ $OD = \sqrt{\overline{OA}^2 - \overline{DA}^2} = \sqrt{25 - 16} = 3.$ $\therefore CD = 4 - 3 = 1.$ $\therefore$ area of zone $BA = 2\pi 5 \times 1 = 31\frac{3}{7}$. *Ans.*

19. $2\pi rh = 2\pi r'h'$. $\therefore rh = r'h'$. $\therefore V : V' = \pi r^2 h : \pi r'^2 h' = (rh)r : (r'h')\,r' = r : r'$.

20.

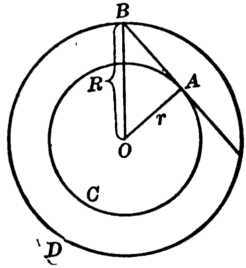


Let $ABCD$ be the quadrilateral in space with diagonals BD and AC. Let the midpoints of AD, AC, BC, BD, be P, Q, R, S respectively.

Then in the $\triangle\, ADC$, $PQ \parallel DC$ and $= \frac{1}{2}\, DC$ (§§ 300, 304). Also in the $\triangle\, BDC$, $SR \parallel DC$ and $= \frac{1}{2}\, DC$ (§§ 300, 304). $\therefore PQ = SR$ (Ax. 1). $PQ \parallel SR$ (§ 464). $\therefore PQRS$ is ▱ (§ 161).

21. Denote the no. of ft. in the inside dimensions of the box by x, $2x$, $4x$. To get the outside dimensions, 2×2 in. or $\frac{1}{3}$ ft. must be added to each of the inside dimensions. $x \cdot 2x \cdot 4x = 216$ cu. ft. $\therefore x = 3$ ft. $3\frac{1}{3} \times 6\frac{1}{3} \times 12\frac{1}{3}$ cu. ft. $-$ 216 cu. ft. $= 44\frac{10}{27}$ cu. ft., or $532\frac{4}{9}$ lumber ft. *Ans.*

22. Let OAC be the smaller, and OBD the larger sphere. The section of the larger sphere made by a plane tangent to the smaller



sphere at A will be a circle with radius AB (§ 641). $AB \perp OA$ (§ 210). $\therefore \overline{AB}^2 = R^2 - r^2$. $\therefore$ area of section $= \pi(R^2 - r^2)$. *Ans.*

23. For figure see Ex. 18, p. 385. Let $DC = b_1$, $dc = b_2$. $\therefore QT = \frac{1}{2}b_1$, $PR = \frac{1}{2}b_2$. $\therefore ST = \frac{1}{2}(b_1 - b_2$, $RS = h$. $\therefore RT = l = \sqrt{\frac{1}{4}(b_1 - b_2)^2 + h^2} = \frac{1}{2}\sqrt{(b_1 - b_2)^2 + 4h^2}$.

24. Draw a diagonal and use § 686.

25. $a(p + q + r)$.

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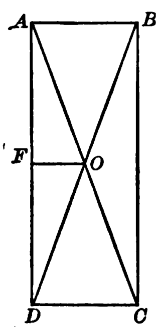
26. Denote the altitude of the given zone by x. Then the rest of the surface of the sphere is a zone whose altitude is $2r - x$. $\therefore 2\pi r(2r - x) : 2\pi rx = 2\pi rx : 4\pi r^2$, or $2r - x : x = x : 2r$. $\therefore x = r(\sqrt{5} - 1)$. *Ans.*

27. See § 578. $1 : a^2$ and $1 : a^3$. *Ans.*

28. $81\pi x = 60 \times 231$ cu. in. $\therefore x = 54\frac{4}{9}$ in. *Ans.*

29.

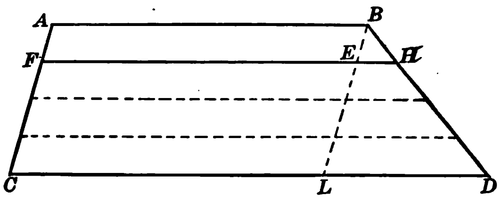


Let AD be the side about which the rectangle $ABCD$ rotates as an axis. Then $AB = r$, $AD = h$. Draw $OF \perp AD$. Then by

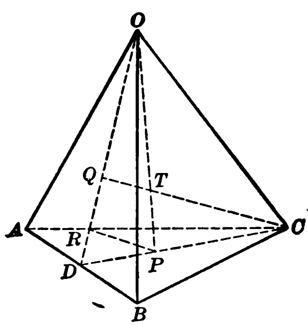
similar ▲, $OF = \frac{1}{2}AB = \frac{1}{2}r$. $\therefore V = \pi r^2 h = \pi r(rh) = 2\pi\frac{r}{2}(rh) =$ (circf. made by O) $\times$ (area rectangle $ABCD$).

30. By § 320, $h : 5$ in. $= 5$ in. $: 16\frac{2}{3}$ in. $\therefore h = \frac{3}{2}$ in. $\therefore S$ of zone $= 78\frac{4}{7}$ sq. in. *Ans.*

31.



The demonstration of Ex. 31 depends on the theorem that if transversals $\parallel$ to the bases of the trapezoid divide each leg into n equal parts, then $(n - 1)$ times one base + the other base $= n$ times the transversal nearest the former base. Thus let $ABCD$ be a trapezoid and let 3 transversals divide AC, and BC into 4 equal parts. Let FH be nearest AB, and draw $BL \parallel AC$, cutting FH in E. Then $LD = 4\,EH$ (by similar ▲). $\therefore 3\,AB + CD = 3\,AB + CL + LD = 3\,AB + AB + 4\,EH = 4\,FE + 4\,EH = 4\,FH$. Let O–ABC be the given tetrahedron. Drop ⊥s from A, B, C, D, P, T to some plane and denote these ⊥s by p_1, p_2, p_3, x, y, z. Then x (the $\perp$ from

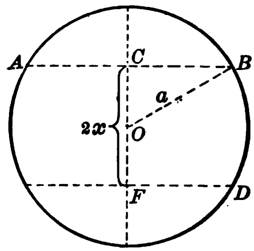


$D) = \frac{p_1 + p_2}{2}$. y (the $\perp$ from $P) = \frac{1}{3}\left[2\left(\frac{p_1 + p_2}{2}\right) + p_3\right] = \frac{p_1 + p_2 + p_3}{3}$. $OT = 3\,TP$. For T is equidistant from A, B, C,

and D (§ 654). $\therefore$ pyramid $TABC = \frac{1}{4}O\text{-}ABC$. z (the $\perp$ from T) $= \frac{1}{4}\left[3\left(\frac{p_1 + p_2 + p_3}{3}\right) + p_4\right] = \frac{p_1 + p_2 + p_3 + p_4}{4}$.

32. Use §§ 677, 687.

33.



Let O be the given sphere and BD the arc of the zone. Denote CF, the altitude of the zone, by $2x$. Then CB, the radius of one base of the zone, $= \sqrt{a^2 - x^2}$. $\therefore S$ of zone $= 2\pi a(2x) = 4a\pi x$. Sum of the bases $= 2\pi(a^2 - x^2)$. $\therefore x = 2(\sqrt{2} - 1)a$. *Ans.*

34. The demonstration is best made by the use of a material (as pasteboard) tetrahedron. It will be found that in general the plane divides the tetrahedron into two polyhedrons, the external faces of each of which consist of two quadrilaterals and two triangles. Denoting these by a, b, c, d, and a', b', c', d' respectively, then $a = a'$, $b = b'$, $c = c'$, $d = d'$, by equality of sides and $\angle$s. Hence by § 509, the corresponding trihedral $\angle$s in the two solids are equal, and the polyhedrons may be made to coincide.

35. 1590.28+ sq. in.

36. First, by the method of § 134, prove that in any spherical $\triangle$, if two sides are unequal, the angles opposite them are unequal, and the greater angles opposite the greater side. Then in the figure, p. 449 (of textbook) $\angle A > \angle B > \angle C$. Hence by § 680, $a' < b' < c'$.

37. 27 times.

38. See Ex. 26, p. 475.

ed

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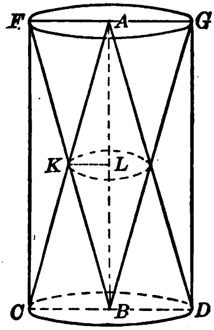
1. Pass a plane through AB, the common axis of the two cones. This plane will intersect the given conical surfaces in the elements AC



and BF. Since FA = and $\parallel CD$, AC and BF are the diagonals of a rectangle and hence bisect each other at K. Draw $KL \perp AB$. Hence by similar $\triangle$, $KL = \frac{1}{2} FA = \frac{1}{2} r$.

2. The volume inclosed by the outer surface of the spherical shell is double that of the original sphere. Denoting these volumes by V and V', and the radius of the outer surface of the shell by x,
$V : V' = 1 : 2 = 12^3 : x^3$.
$\therefore x = 12 \sqrt[3]{2}$ in. $= 15.118+$ in.
$15.118+$ in. $- 12$ in. $= 3.118+$ in. *Ans.*

3. 8 in.

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1. $\pi \left(\frac{r_1 + r_2}{2}\right)^2 h + \frac{\pi}{3}\left(\frac{r_1 - r_2}{2}\right)^2 h = \frac{1}{3}\pi h\,(r_1^2 + r_2^2 + r_1 r_2)$.

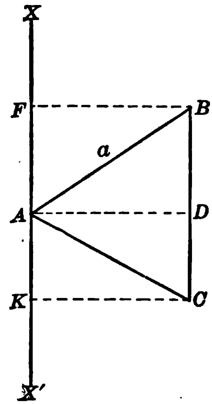
2. $S = 4\pi a^2 \sqrt{2}$, $V = \pi a^3 \sqrt{2}$.

3. The angle between the axis of the fixed cone and that of the rolling cone is constant. Hence the locus is the surface of a cone of revolution (Ex. 10, p. 423).

4. The altitude of the $\triangle = \frac{a\sqrt{3}}{2}$, hence r of the inscribed circle $=$ $\frac{a\sqrt{3}}{6}$, and R of the circumscribed circle $= \frac{a\sqrt{3}}{3}$. $\therefore S = \frac{\pi a^2}{3}$, $V = \frac{\pi a^3 \sqrt{3}}{54}$. $S' = \frac{4\pi a^3}{3}$, $V' = \frac{4\pi a^3\sqrt{3}}{27}$. *Ans.*

5. Let ABC be the given equilateral $\triangle$, with $BC = a$. Let XX' be the axis of rotation. $AD = \frac{a\sqrt{3}}{2} = FB = KC$. AB and AC each

describe the surface of a cone having AF for altitude and FB for radius of base. $\therefore$ area $AB = \pi\left(\frac{a\sqrt{3}}{2}\right)a =$ area AC. Area $BC =$



$2\pi\left(\frac{a\sqrt{3}}{2}\right)a$. Total area $= 2\pi a^2\sqrt{3}$. *Ans.* The volume = vol. of cylinder described by $FBCK$ − vol. of the two cones described by AFB and $ACK = \pi\left(\frac{a\sqrt{3}}{2}\right)^2 a - 2\frac{\pi}{3}\left(\frac{a\sqrt{3}}{2}\right)^2\frac{a}{2} = \frac{\pi a^3}{2}$. *Ans.*

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6. Conceive the frustum of a triangular pyramid constructed in which the areas of the lower and upper bases of the frustum are equivalent to the areas of the lower and upper bases of the given frustum of a cone, each pair of corresponding bases being in the same plane. Complete the pyramid and cone of which the frustums are parts. Prove a property for the cone similar to that proved for the pyramid in § 566, etc.

7. If an edge of the tetrahedron be denoted by a, the radius of the sphere by r, and the altitude of the tetrahedron by h, $h = \frac{a\sqrt{6}}{3}$ (see Ex. 9, p. 329). But radii drawn from the center of the sphere to the vertices of the tetrahedron divide the given tetrahedron

into four equal tetrahedrons. $\therefore$ r of sphere $= \frac{3}{4}h = \frac{3}{4}\left(\frac{a\sqrt{6}}{3}\right) =$ $\frac{a\sqrt{6}}{4}$.

$\therefore 4r = a\sqrt{6}$. $\therefore a : r = 4 : \sqrt{6}$.

Hence r being given, a can be constructed, etc.

9. $V = \frac{1}{6}h\,(b_1 + b_2 + 4m)$.
$= \frac{1}{6}h\,(b + b + 4b)$.
$= hb$.

10. In the prismatoid formula, let $b_2 = 0$, $b_1 = b$, $m = \frac{1}{4}b$. $\therefore V = \frac{1}{6}h\,(b + b) = \frac{1}{3}hb$.

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11. $\therefore b_1 + b_2 + 2\sqrt{b_1 b_2} = 4m$.
$\therefore V = \frac{1}{6}h\,(b_1 + b_2 + 4m)$,
becomes $V = \frac{1}{6}h\,(b_1 + b_2 + b_1 + b_2 + 2\sqrt{b_1 b_2})$.
or $V = \frac{1}{3}h\,(b_1 + b_2 + \sqrt{b_1 b_2})$.